# ANALYSIS OF CONCRETE SLABS ON GROUND AND SUBJECTED TO WARPING AND MOVING LOADS

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by

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Joint Highway Research Project

PURDUE UNIVERSITY
LAFAYETTE INDIANA

Final Report

ANALYSIS OF CONCRETE SLABS ON GROUND SUBJECTED
TO WARPING AND MOVING LOADS

TO: G. A. Leonards, Director
Joint Highway Research Project

June 20, 1967

File: 5-7-4

FROM: H. L. Michael, Associate Director
Joint Highway Research Project

Project: C-36-63D

The attached Final Report "Analysis of Concrete Slabs on Ground and Subjected to Warping and Moving Loads" by [illegible] H. Lewis is presented to the Board for the record. Mr. Lewis also used the report as his Ph.D. Thesis. The research and report have been conducted under the direction of Professor M.E. Harr of the Civil Engineering faculty.

The research has developed a theory whereby stresses and deflections can be calculated for a series of rectangular slabs lying on a visco-elastic foundation and subjected to a moving load. Two problems of pavement slabs - partial support caused by warping and a reduction of subgrade support over some narrow region - are considered in the research report.

The report is presented to the Board in fulfillment of the Plan of Study approved October 31, 1963.

Respectfully submitted,

Harold L. Michael

Harold L. Michael,
Associate Director

HLM:jgs

Attachment

| | | |
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| Copy: F. L. Ashbaucher | C. E. Harvey | M. B. Scott |
| W. L. Dolch | J. F. McLaughlin | H. T. Spencer |
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Final Report

# ANALYSIS OF CONCRETE SLABS ON GROUND SUBJECTED TO WARPING AND MOVING LOADS

by

Karl H. Lewis
Graduate Instructor in Research

Joint Highway Research Project

File No.: 9-7-4

Project No.: C-36-63D

Purdue University

Lafayette, Indiana

June 20, 1967

# ACKNOWLEDGMENTS

The writer wishes to express his gratitude to the Joint Highway Research Project which sponsored this study. He is deeply appreciative of the assistance of his major professor Dr. M. E. Harr. Without his inspiration, support and encouragement, the completion of the work would not have been possible.

The author also wishes to express his gratitude to Dr. G. A. Leonards, Head of the School of Civil Engineering and to Dr. J. F. McLaughlin, Assistant Head of the School of Civil Engineering, Purdue University.

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# ABSTRACT

Lewis, Karl Harcourt. Ph.D., Purdue University, June 1967. "Analysis of Concrete Slabs on Ground and Subjected to Warping and Moving Loads." Major Professor: M. E. Harr.

A theory has been developed whereby stresses and deflections could be calculated for a series of rectangular slabs lying on a viscoelastic foundation and subjected to a moving load. The stresses and deflections are caused by the weight of the slab, the moving concentrated load, and by linear temperature (or moisture) variations that cause sufficient warping so that the slab is only partially supported by its foundation.

Part I considers the problem of partial support caused by warping (in this thesis, the temperature at the top of the slab is smaller than that at the bottom), while Part II concerns itself with the effect of a reduction in subgrade support over some narrow region.

In both parts, the support conditions were simulated by a Kelvin viscoelastic model, and zones (which depended on the value of subgrade reaction) were set up so that the solutions to the governing differential equations could be reduced to a set of simultaneous algebraic equations. For the problem studied in Part I, the resulting simultaneous equations were non-linear and a method of functional iteration equivalent to an N-dimensional Newton's Method was used. In the case of Part II, the equations were linear and the Method of Crout Reduction was used. In both parts, the equations were solved with the aid of an IBM 7090 digital computer using a Fortran source program.

It was found that when partial support due to warping exists, the tensile stress in the slab can increase with increasing velocity of load. Moreover, the maximum deflection (downward) need not occur when the velocity of the load is equal to zero. The reduction in subgrade support over a narrow region (approximately 8 feet or less) does lead to deflections and stresses which are higher than those calculated using the initial value of subgrade reaction. This is particularly evident when the load is over the region of reduced subgrade reaction.

# INTRODUCTION

Improvements in the performance of concrete pavements for highways and airports have for many years been the cause of concern among Civil Engineers interested in these problems. This concern has not only been due to the ever increasing cost of construction and maintenance, but also to the preponderance of cracks which exist in pavements.

The development of these cracks depends on several factors, (such as type of subgrade, deterioration of the concrete, temperature effects and load) (68)* and, although they may or may not represent a failed condition, they do indicate deficiencies in the analysis, design or construction of highway pavements. Cracks not only detract from the general appearance of a highway, but often they lead to driving discomfort and pavement deterioration; especially when load transfer and subgrade support are lost.

As far as the construction of concrete pavements is concerned, advancement in the development of the principles of soil mechanics has made great strides in limiting the use of poor subgrades and has virtually eliminated the use of soils susceptible to frost action and pumping. In addition, specifications (1, 63) geared at promoting the use of sound concrete and the practice of good construction methods, have become widespread.

* Figures in parenthesis refer to references in Bibliography.

Along with refinements in construction procedures, methods of monitoring the factors influencing concrete pavements have been developed (21, 22), and strains as well as deflections of pavements can be determined for both static and dynamic loadings (16, 24, 54). Temperature differences between the upper and lower faces of the slab can be measured with the use of thermo-couples (67), and variations in moisture content can be established from the dielectric properties of concrete (3).

Today, the problem of material weakness has been reduced, if not solved, and significant progress has been made in the ability to build pavements and to measure their properties which seem important; however, only little has been done to upgrade the methods of analysis. Current procedures for designing and evaluating pavements (68) are still based on static loads and except for introducing equivalent static loadings (2), they do not account for the dynamic response of the pavement to moving loads. Moreover, the pavement is usually assumed to be fully supported at all times even though this often is not the case (16, 29, 30).

The present trend in transportation is towards heavier loads and higher speeds and consequently there is an increasing need to be able to predict performance. This can only be done when more is known about the input and causal factors of traffic loads, curling, and the loss of subgrade support. It is to help satisfy this need, that this thesis is dedicated.

# REVIEW OF LITERATURE

In the early stages of development, pavement design and road construction consisted of a collection of rule of thumb procedures based on empirical knowledge. As early as 1909, test roads were being used to determine the best type of pavement for the prevailing traffic (49). Between 1920 and 1940, much work was done on the classification of soils (5, 57) and Highway Engineers were able to correlate pavement performance with subgrade types.

With the advent of World War II, engineers were faced with the problem of designing pavements for greater wheel loads than previously thought necessary. They realized that they could not afford the long period of time which was required to develop design procedures based on past experiences and thus sought more rational basis of design. This search ultimately led to procedures which form the core of pavement design today.

## Static Load Solutions

In 1884, Hertz (19) first published a solution to the problem of an elastic plate on a Winkler-type foundation (66); however, it was not until the advent of Westergaard's solution in 1926 that a highway pavement was treated in this manner and the problem was approached from a purely theoretical point of view. Today, this work still forms the basis of the analytical bent in pavement design.

Westergaard (64) solved the problem of a slab fully supported on a Winkler foundation and subjected to static loads applied at the interior, free edges and corners of the slab. Later, Kelley (32), Spangler (56), Pickett (45) and Westergaard (65) himself extended these original solutions to account for linear temperature variations. Thomlinson (58) in 1940 further modified Westergaard's solution by assuming a simple harmonic temperature variation which resulted in curved temperature variations throughout the depth of the slab. In 1957, Freudenthal and Lorsch (15) used the three fundamental models (Maxwell, Kelvin and Standard Solid) to study the problem of an infinite beam on a viscoelastic foundation; and in 1958, Hoskin and Lee (26) solved the problem of an infinite plate on a viscoelastic foundation.

The foregoing solutions neglect the shearing forces generated at the pavement-base interface. Several mechanisms have been offered to account for this effect. Filonenko-Borodich (13) considered a set of springs held together by a membrane, while Schiel (55) took a fluid which exhibited surface tension as his soil model. Pasternak (43) and Kerr (34) on the other hand, considered a beam of unit depth resting on a bed of interrelated springs as their foundation, and Pister and Williams (44) used the shear interactions suggested by Reissner (52). Perhaps the most realistic approach offered to account for shear in an elastic base is that given by Klubin. Klubin (35) expressed the pavement reaction by an infinite series of Tschebyscheff polynomials which also accounts for the two elastic constants (Modulus of Elasticity and Poisson's Ratio).

It should be noted that Klubin's solution is in fact an elastic solution while the others which impose Winkler assumptions are not. In spite of this, since the Winkler foundation affords a much simpler

analysis and generally gives good agreement with field data, it will no doubt remain popular.

All of the above analyses are based on the assumption that the slab maintains contact with its support at all points and at all times. Experimental and field studies (16, 29, 30) have shown this assumption to be seriously in error and a few investigators have accounted for the effects of only partial support. In 1959, Leonards and Harr (37) solved the problem of a partially supported slab on a Winkler foundation for linear temperature and/or moisture variations. Later, Reddy, Leonards, and Harr (51) extended this analysis by introducing non-linear temperature variations as well as a viscoelastic foundation. In all of these analytical procedures, only the case of symmetrical, statically applied loads were considered.

## Dynamic Load Solutions

For many years, the problem of determining the stresses and deflections in a vibrating plate has been of interest primarily to mathematicians. Raleigh (48) and Lamb (36) using the classical beam theory developed by Euler (11) and Bernoulli (4), studied several problems dealing with the vibration of bars, membranes and plates. Later, Ritz (53) elaborated on this work and made significant contributions toward the study of vibrating rectangular plates.

Recently, engineers have felt the need to account for the dynamic response of pavements and several solutions have evolved. Pioneering these solutions was the work of Timoshenko (60), Hovey (28) and Ludwig (39) in their studies of the dynamics of rails subjected to moving loads. In 1943, Dorr (10) using Fourier integrals extended the idea to a beam,

but in all of these solutions the foundation was represented by a Winkler model which exhibited no viscous effects.

Later in 1953, Kenney (33) added the effects of linear damping and by means of the method of undetermined coefficients was able to examine the relationship between deflections and critical velocity. Kenney found that at a constant velocity ratio (velocity/critical velocity) the deflection amplification factor (ratio of deflection behind moving load to that under static load) increased as the damping ratio (damping coefficient/critical damping coefficient) decreased. For heavy damping (damping ratio $\approx$ 1.5), the deflection amplification decreased with increasing velocity, however at low damping (damping ratio $\approx$ 0.3) this was only partially true. In the latter case, the deflection amplification increased until about the point of critical velocity (becoming infinite for a damping ratio = 0) and then decreased. In the case of the moment amplification factor (ratio of maximum moment under moving load to that under static load), a similar trend was experienced; however, at low damping ratios, the maximum moment occurred at velocities slightly below the critical value. Mathews (40) in 1958 also studied a similar problem using Fourier transforms. Crandall (9), recognizing the contributions of Timoshenko (60) and Mindlin (41) on shear deformation, extended Kenney's solution by replacing the Bernoulli-Euler beam with the Timoshenko model. Crandall found that whereas the former model had a single resonant frequency and a single critical velocity, the latter had three of each.

In a recent work, Thompson (59) elaborated on Kenney's solution and showed that the solutions for the deflections fall into three distinct regimes. Specifically, Thompson showed that these regimes were defined by the value of the discriminant of the fourth order characteristic

equation. If the value of the discriminant as defined by

$$\Delta = 16 \left[ 4 (1-\epsilon^2) \theta^8 - (8 - 36\epsilon^2 + 27\epsilon^4) \theta^4 + 4 \right] \qquad (1)$$

where $\epsilon$ = the damping ratio

and $\theta$ = the velocity ratio

is greater than zero, the characteristic equation has no real roots. If $\Delta$ is equal to zero, there is one, real, double root; and if $\Delta$ is less than zero there are two, real, unequal roots.

The solutions presented by Thompson demonstrate that at static conditions, the deflection curve is symmetrical (with maximum deflection occurring under the load); but as velocity increases, the point of maximum deflection falls further and further behind the load. Upward deflections occur behind the moving load when $\Delta$ is greater than zero, however when $\Delta$ is less than zero, there are no upward deflections (i.e. the deflected surface does not intersect the axis of zero deflection, but simply approaches it at some great distance behind the load).

Several investigators have also considered the road loading system as an interaction between two major components which are interdependent, namely vehicles and roads. Fabian, Clark and Hutchinson (12) examined the elements of each component and developed some basic mathematical models. Analysis of their vehicle sub-system showed that the magnitude of the dynamic load is a function of vehicle dynamic properties and apparent road profile, and may in fact be significantly greater than the static load. Quinn and De Vries (46) used an experimental procedure to determine the highway and vehicle characteristics and showed that if these quantities are known, the reaction of a vehicle on a highway can be predicted.

Recognizing the difficulty in determining appropriate characterizations for both highway and vehicle, Sanborn (54), in a recent work used a more direct approach. He simultaneously measured the dynamic load applied to a pavement by a moving vehicle, and the dynamic stresses and displacements induced by that load in the pavement and its subgrade. The displacements and stresses within the pavements studied varied with velocity, but no definite pattern was established. However, dynamic variations of load, for the load and range of velocity covered (4500 lbs. and 0 to 60 mph respectively), appeared to have little effect on pavement response.

In general, most of the analytical studies reported in the literature on moving loads on pavements (68), do not account for the interaction between vehicle and road. For the most part, these reports show that for fully supported pavements, the lower the velocity, the greater the deflections and stresses in the slab. Measurements of displacement (21, 23, 24) generally support this finding; however, significant data (16) does exist to suggest the possibility that displacements may increase at velocities greater than 30 to 40 mph. Also to be noted is the fact that tests on tire forces on pavements (47) do indicate that the "average" force produced on the pavement by a moving load, increases with velocity.

Obviously, there are several interdependent factors such as the vehicle's suspension system, tire pressures and road profile which should be included in the analysis of pavements; however at the present time, the interaction between vehicle and pavement is not well understood. In spite of this, further insight into the highway problem can be gained by examining some of the simplifying assumptions which appear in current

analyses of highway pavements. Thus, it is the object of this thesis to obtain by analytical means the stresses and deflections in a partially supported concrete slab when subjected to a moving load. Such partial support may be caused by temperature and moisture gradients or by the weakening and partial or complete loss of subgrade.

## FORMULATION OF PROBLEM

In the first part of this thesis, the problem is considered of a slab lying on a foundation while subjected to moving loads as well as to temperature (and/or moisture) gradients which would cause upward curling.* The second part will investigate conditions where a weakening of the subgrade has developed. The conditions in Part II may exist where (a) water has infiltrated under the pavement from the side of the roadway; or (b) where a joint has opened or where a crack has occurred in a warped pavement to such an extent as to enable it to regain complete contact with the foundation. When such an opening exists, water may infiltrate through the surface of the pavement. Under both conditions (a) and (b), the infiltration may weaken the subgrade and may eventually lead to pumping. The following assumptions are made for both parts in order to render them tractable:

* Upward curling is understood to exist when temperature gradients cause the ends of the slab to rise vertically.

# ASSUMPTIONS

1. A highway pavement can be represented by an array of rectangular plates.
2. The usual assumptions of plate theory hold, namely: the plate is homogeneous, isotropic and obeys Hooke's law; deflections are small in comparison to thickness; plane cross sections normal to the middle plane in the unstressed condition remain normal to this surface after bending; the effects of rotatory inertia and shear deformation can be neglected.
3. The highway base material acts like a set of linear viscoelastic elements. The inertia of the material is neglected.
4. External forces acting on the plate are those due to a constant-velocity line load and gravity.
5. The plates are subjected to changes in temperature (and/or moisture) which vary linearly with depth. The variation in temperature is constant on all planes parallel to the upper and lower plate surfaces and is independent of time.

PART I

In addition to the foregoing assumptions, further assumptions regarding the interaction between slabs are required. For the case studied here, it is assumed that no bending moment exists between slabs (this will be discussed later). However, some load transfer can be accounted for by specifying a shearing force between slabs, equivalent to that existing in an infinite slab.

In general, the governing differential equation describing the pavement section illustrated in Figure 1 can be expressed as follows (42):

$$D\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4 w}{\partial y_1^4}\right) + \rho H \frac{\partial^2 w}{\partial t^2} = q(x_1, y_1, t) - p(x_1, y_1, t) \quad (2)$$

where $D$ = flexural rigidity of the slab

$w$ = mid-plane deflection of the slab (positive downward)

$x_1, y_1$ = fixed coordinates

$\rho$ = density of the slab

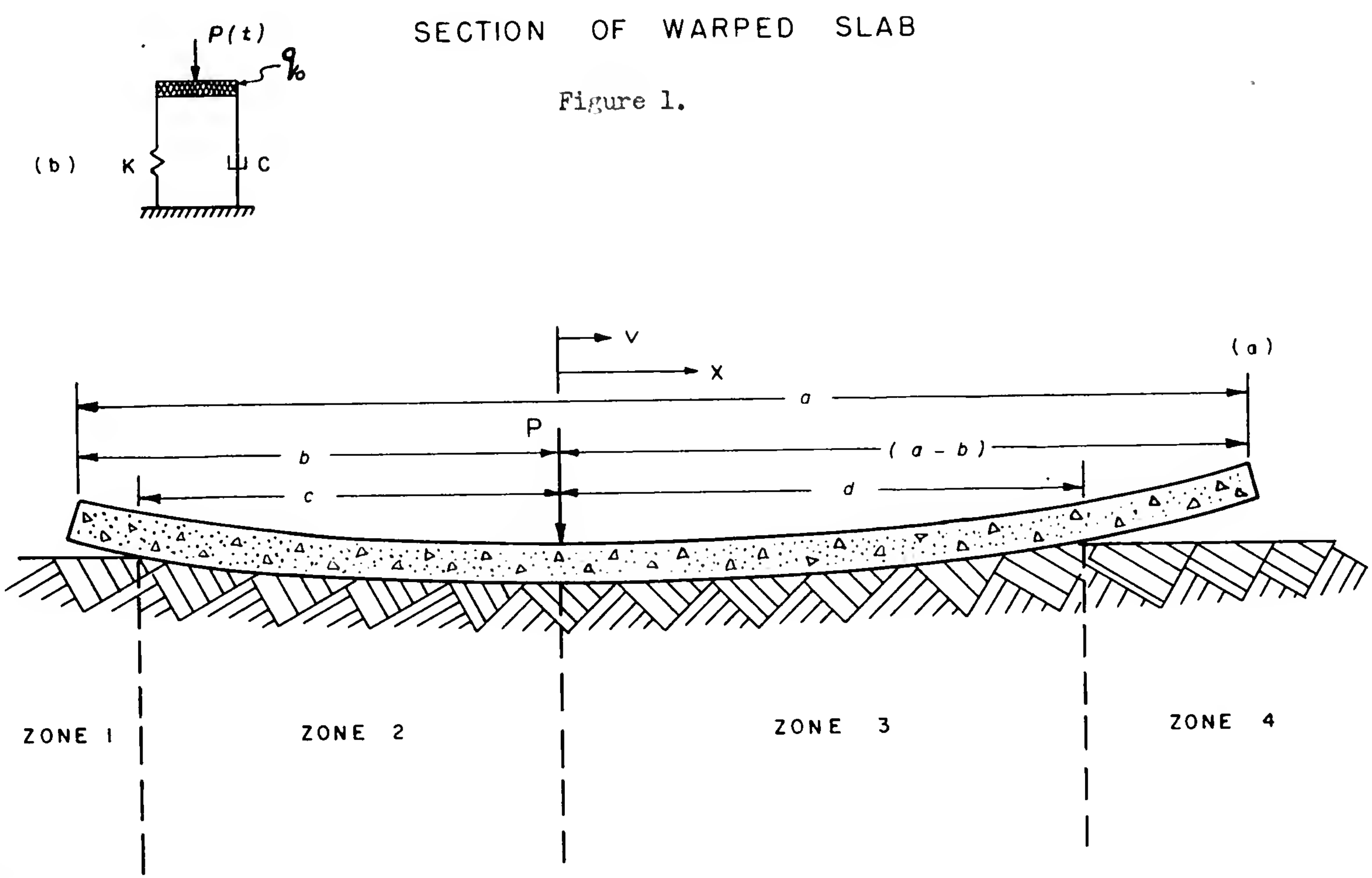
$H$ = slab thickness

$q$ = surface loading

$p$ = foundation reaction

$t$ = time

Using Assumption 3, the foundation reaction, $p$, may be expressed as (see Figure 1b): $p(x_1, y_1, t) = C\frac{\partial w}{\partial t} + Kw$, where $C$ is the damping coefficient and $K$ is the modulus of subgrade reaction. Applying the loading conditions implied by Assumption 4, and assuming that the




SECTION OF WARPED SLAB

Figure 1.

deflection does not vary in the $y_1$ direction, Equation 2, for a constant cross section of pavement, becomes

$$D \frac{\partial^4 w}{\partial x_1^{\,4}} + \rho H \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + Kw = q_o + P(x_1, t) \tag{3}$$

in which $q_o$ = unit weight of slab

and $P$ = the moving line load

Employing the transformation, $x = (x_1 - vt)$, the equation is seen to reduce to a function of only the one variable, x. Now if in addition the moving load, P, is introduced with the boundary conditions, the following differential equations are obtained:

For zones 1 and 4

$$D \frac{d^2 w}{dx^4} + \rho Hv^2 \frac{d^2 w}{dx^2} = q_o \tag{4a}$$

For zones 2 and 3

$$D \frac{d^4 w}{dx^4} + \rho Hv^2 \frac{d^2 w}{dx^2} - Cv \frac{dw}{dx} + Kw = q_o \tag{4b}$$

The boundary conditions for the indicated zones may be summarized as follows: (see Appendix I for nomenclature)

$$\begin{aligned}
&(a) \quad M_1(-b) = 0 \\
&(b) \quad V_1(-b) = \overline{V}\,(-b) \\
&(c) \quad w_1(-c) = 0 \\
&(d) \quad w_2(-c) = 0 \\
&(e) \quad w_1'(-c) = w_2'(-c) \\
&(f) \quad w_1''(-c) = w_2''(-c) \\
&(g) \quad w_1'''(-c) = w_2'''(-c) \\
&(h) \quad w_2(0) = w_3(0) \\
&(i) \quad w_2'(0) = w_3'(0) \\
&(j) \quad w_2''(0) = w_3''(0) \\
&(k) \quad w_3'''(0) - w_2'''(0) = P/D
\end{aligned} \tag{5}$$

(l) $w_3(d) = 0$

(m) $w_4(d) = 0$

(n) $w_3'(d) = w_4'(d)$

(o) $w_3''(d) = w_4''(d)$

(p) $w_3'''(d) = w_4'''(d)$

(q) $M_4(a-b) = 0$

(r) $V_4(a-b) = \overline{V}\,(a-b)$

## Solution of Problem

For the case where $\Delta > 0$ (Equation 1), and the warped slab is as represented by Figure 1, the deflections and stresses in the slab are as follows: (see Appendix II for derivation)

For $-b \leq x \leq -c$

$$w_1 = A_1 + A_2 x + A_3 \cos mx + A_4 \sin mx + \frac{q_o}{2D}\left(\frac{x}{m}\right)^2 \tag{6}$$

$$\sigma_1 = \frac{-6\rho v^2}{H}\left[A_3(\cos mb - \cos mx) - A_4(\sin mb + \sin mx)\right] \tag{7}$$

For $-c \leq x \leq 0$

$$w_2 = q_o/K + e^{-a_1 x}(B_1 \sin b_2 x + B_2 \cos b_2 x) + e^{a_1 x}(B_3 \sin b_1 x + B_4 \cos b_1 x) \tag{8}$$

$$\sigma_2 = -\frac{6D}{H^2}\Bigg[e^{-a_1 x}\Big(B_1\left[(a_1^2 - b_2^2)\sin b_2 x - 2a_1 b_2 \cos b_2 x\right] + B_2\left[2a_1 b_2 \sin b_2 x + (a_1^2 - b_2^2)\cos b_2 x\right]\Big) + e^{a_1 x}\Big(B_3\left[(a_1^2 - b_1^2)\sin b_1 x + 2a_1 b_1 \cos b_1 x\right] + B_4\left[(a_1^2 - b_1^2)\cos b_1 x - 2a_1 b_1 \sin b_1 x\right]\Big) + \beta(1+\mu)\frac{\Delta T}{H}\Bigg] \tag{9}$$

For $0 \leq x \leq d$

$$w_3 = q_o/K + e^{-a_1x}(C_1 \sin b_2x + C_2 \cos b_2x) + e^{a_1x}(C_3 \sin b_1x + C_4 \cos b_1x) \tag{10}$$

$$\sigma_3 = \frac{-6D}{H^2}\Big[ e^{-a_1x} \Big(C_1[(a_1^2 - b_2^2) \sin b_2x - 2a_1b_2 \cos b_2x] + C_2[2a_1b_2 \sin b_2x + (a_1^2 - b_2^2) \cos b_2x]\Big) + e^{a_1x}\Big(C_3[(a_1^2 - b_1^2) \sin b_1x + 2a_1b_1 \cos b_1x] + C_4[(a_1^2 - b_1^2) \cos b_1x - 2a_1b_1 \sin b_1x]\Big) + \beta(1+\mu)\frac{\Delta T}{H}\Big] \tag{11}$$

For $d \leq x \leq (a-b)$

$$w_4 = D_1 + D_2x + D_3 \cos mx + D_4 \sin mx + \frac{q_o}{2D}\left(\frac{x}{m}\right)^2 \tag{12}$$

$$\sigma_4 = \frac{-6\rho v^2}{H}\Big[D_3[\cos m(a-b) - \cos mx] + D_4[\sin m(a-b) - \sin mx]\Big] \tag{13}$$

Applying the conditions listed in Equation 5, to Equations 6, 8, 10 and 12, the following set of non-linear equations are obtained.

$$A_1 - A_2c + A_3 \cos mc - A_4 \sin mc = -\frac{q_o}{2D}\left(\frac{c}{m}\right)^2 \tag{14}$$

$$e^{a_1c}(B_2 \cos b_2c - B_1 \sin b_2c) + e^{-a_1c}(B_4 \cos b_1c - B_3 \sin b_1c) \doteq -q_o/K \tag{15}$$

$$e^{a_1 c}[B_1(b_2 \cos b_2 c + a_1 \sin b_2 c) - B_2(a_1 \cos b_2 c - b_2 \sin b_2 c)] +$$

$$e^{-a_1 c}[B_3(b_1 \cos b_1 c - a_1 \sin b_1 c) + B_4(a_1 \cos b_1 c + b_1 \sin b_1 c)] -$$

$$A_2 - m(A_3 \sin mc + A_4 \cos mc) = -\frac{q_o c}{m^2 D} \tag{16}$$

$$e^{a_1 c}\Big[B_2\left[(a_1^2 - b_2^2) \cos b_2 c - 2a_1 b_2 \sin b_2 c\right] - B_1\left[(a_1^2 - b_2^2) \sin b_2 c +\right.$$

$$\left.2a_1 b_2 \cos b_2 c\right]\Big] + e^{-a_1 c}\Big[B_3\left[2a_1 b_1 \cos b_1 c - (a_1^2 - b_1^2) \sin b_1 c\right] +$$

$$B_4\left[(a_1^2 - b_1^2) \cos b_1 c + 2a_1 b_1 \sin b_1 c\right]\Big] + m^2(A_3 \cos mc - A_4 \sin mc)$$

$$= \frac{q_o}{m^2 D} \tag{17}$$

$$e^{a_1 c}\Big[B_2\left[(3a_1 b_2^2 - a_1^3) \cos b_2 c - (b_2^3 - 3b_2 a_1^2) \sin b_2 c\right] -$$

$$B_1\left[(3a_1 b_2^2 - a_1^3) \sin b_2 c + (b_2^3 - 3b_2 a_1^2) \cos b_2 c\right]\Big] +$$

$$e^{-a_1 c}\Big[B_3\left[(3a_1 b_1^2 - a_1^3) \sin b_1 c - (b_1^3 - 3b_1 a_1^2) \cos b_1 c\right] -$$

$$B_4\left[(b_1^3 - 3b_1 a_1^2) \sin b_1 c + (3a_1 b_1^2 - a_1^3) \cos b_1 c\right]\Big] +$$

$$m^3(A_3 \sin mc + A_4 \cos mc) = 0 \tag{18}$$

$$B_2 + B_4 - C_2 - C_4 = 0 \tag{19}$$

$$b_2(B_1 - C_1) + a_1(C_2 - C_4 - B_2 + B_4) + b_1(B_3 - C_3) = 0 \tag{20}$$

$$2a_1 b_2 (C_1 - B_1) + (a_1^2 - b_2^2)(B_2 - C_2) + 2a_1 b_1 (B_3 - C_3) +$$

$$(a_1^2 - b_1^2)(B_4 - C_4) = 0 \tag{21}$$

$$(b_2^3 - 3b_2a_1^2)(B_1 - C_1) - (3a_1b_2^2 - a_1^3)(B_2 - C_2) +$$

$$(b_1^3 - 3b_1a_1^2)(B_3 - C_3) + (3a_1b_1^2 - a_1^3)(B_4 - C_4) = P/D \tag{22}$$

$$e^{-a_1 d}(C_1 \sin b_2 d + C_2 \cos b_2 d) + e^{a_1 d}(C_3 \sin b_1 d + C_4 \cos b_1 d)$$

$$= - q_o/K \tag{23}$$

$$D_1 + D_2 d + D_3 \cos md + D_4 \sin md = \frac{q_o}{2D}\left(\frac{d}{m}\right)^2 \tag{24}$$

$$e^{-a_1 d}\left[C_1(b_2 \cos b_2 d - a_1 \sin b_2 d) - C_2(a_1 \cos b_2 d + b_2 \sin b_2 d)\right] +$$

$$e^{a_1 d}\left[C_3(a_1 \sin b_1 d + b_1 \cos b_1 d) + C_4(a_1 \cos b_1 d - b_1 \sin b_1 d)\right]$$

$$- D_2 + m(D_3 \sin md - D_4 \cos md) = \frac{q_o d}{m^2 D} \tag{25}$$

$$e^{-a_1 d}\Big[C_1\left[(a_1^2 - b_2^2)\sin b_2 d - 2a_1b_2 \cos b_2 d\right] + C_2\left[2a_1b_2 \sin b_2 d +\right.$$

$$\left.(a_1^2 - b_2^2)\cos b_2 d\right]\Big] + e^{a_1 d}\Big[C_3\left[(a_1^2 - b_1^2)\sin b_1 d + 2a_1b_1 \cos b_1 d\right]$$

$$+ C_4\left[(a_1^2 - b_1^2)\cos b_1 d - 2a_1b_1 \sin b_1 d\right]\Big] + m^2(D_3 \cos md +$$

$$D_4 \sin md) = \frac{q_o}{m^2 D} \tag{26}$$

$$e^{-a_1 d}\Big[C_1\left[(3a_1b_2^2 - a_1^3)\sin b_2 d - (b_2^3 - 3b_2a_1^2)\cos b_2 d\right] +$$

$$C_2\left[(3a_1b_2^2 - a_1^3)\cos b_2 d + (b_2^3 - 3b_2a_1^2)\sin b_2 d\right]\Big] + e^{a_1 d}\Big[C_4\left[(b_1^3 -\right.$$

$$\left.3b_1a_1^2)\sin b_1 d - (3a_1b_1^2 - a_1^3)\cos b_1 d\right] - C_3\left[(3a_1b_1^2 - a_1^3)\sin b_1 d +\right.$$

$$\left.(b_1^3 - 3b_1a_1^2)\cos b_1 d\right]\Big] - m^3(D_3 \sin md - D_4 \cos md) = 0 \tag{27}$$

where $$A_3 = \frac{1}{m^2}\left[\frac{\overline{V}(-b)\sin(mb)}{mD} + \left(\frac{q_o}{m^2 D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\cos(mb)\right] \tag{28a}$$

$$A_4 = \frac{1}{m^2}\left[\frac{\overline{V}(-b)\cos(mb)}{mD} - \left(\frac{q_o}{m^2 D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\sin(mb)\right] \tag{28b}$$

$$D_3 = \frac{-1}{m^2}\left[\frac{\overline{V}(a-b)}{mD}\sin m(a-b) - \left(\frac{q_o}{m^2 D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\cos m(a-b)\right] \tag{28c}$$

and $$D_4 = \frac{1}{m^2}\left[\frac{\overline{V}(a-b)}{mD}\cos m(a-b) + \left(\frac{q_o}{m^2 D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\sin m(a-b)\right] \tag{28d}$$

To solve for the constants in Equations 14 through 27, an initial estimate of the solution was made using the fully supported slab, then a method of functional interation equivalent to an N-dimensional Newton's Method was used with an IBM 7090 computer (see Appendix III for programs). With the constants determined, the stresses were then obtained from Equations 7, 9, 11 and 13. Using this same procedure, the other deflection patterns resulting from the moving load were analyzed (see Figure 2) and the stresses were determined.

## Results

Due to the large number of variables involved, it is impractical to present the results for all considered cases. Instead, numerical results are given for a few typical cases to illustrate general trends and for purposes of comparison with results previously obtained by others. Typical programs are available in Appendix III and they may be used (or modified) to obtain other values desired.

Using the left edge of the slab, as origin, the moving load was considered at intervals of 4 ft. and stresses and deflections were determined for conbinations of the following data.

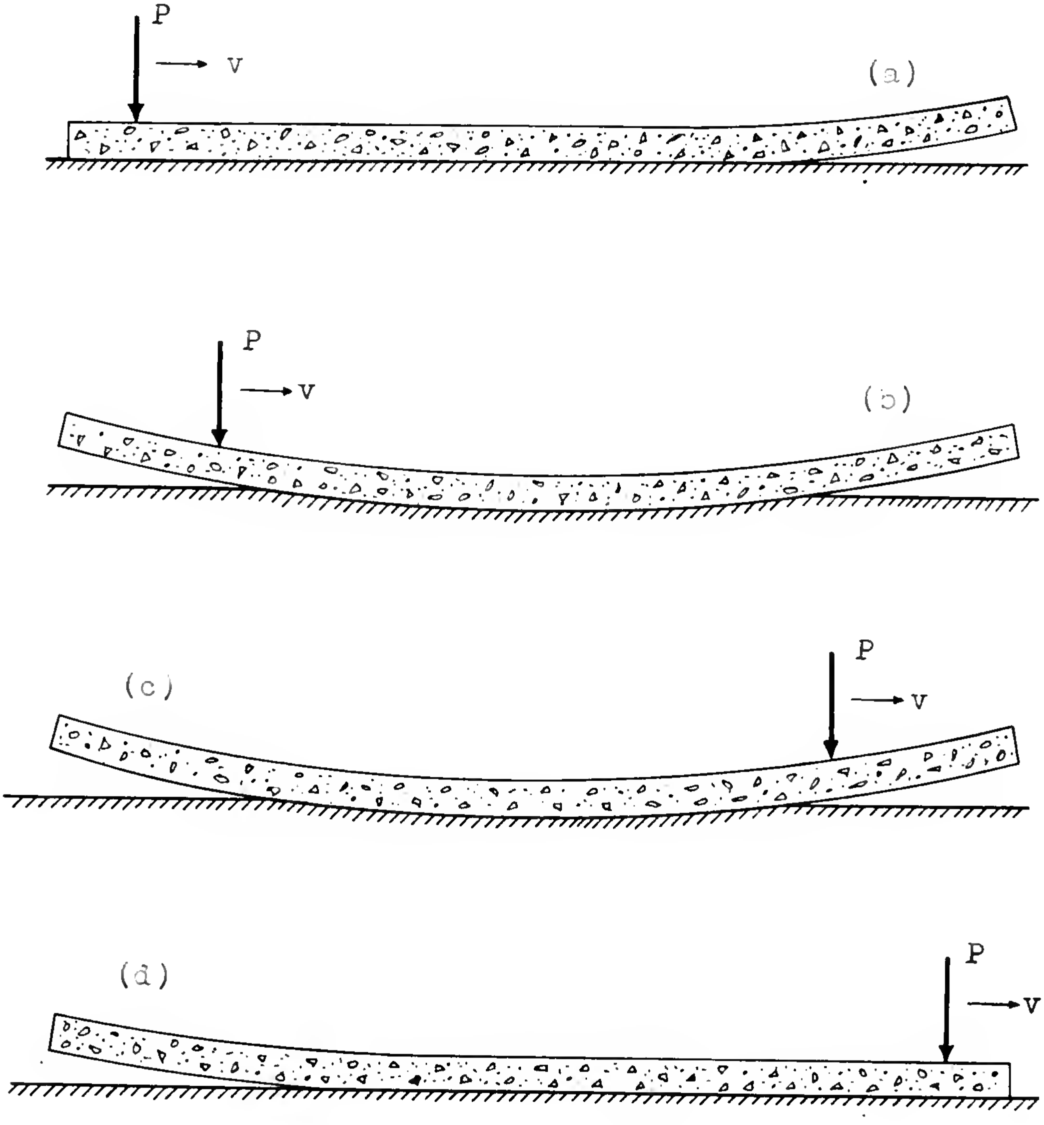



Figure 2. Typical Deflection Patterns

$\mu = 0.15$

$E = 4 \times 10^6$ psi.

$a = 240$ ins.

$H = 8, 10, 12$ ins.

$K = 100, 200$ pci.

$C_r = 1.5, 2.0$

$v = 0, 20, 40, 60, 80$ mph.

$P = 125$ lbs/in. (The value of P was determined using an axle load of 18,000 lbs. over a pavement 12 ft. wide)

$q_o = 150.9$ pcf.

$\Delta T = 20, 30, 40$ deg. F.

$\beta = 6 \times 10^{-6}$ inches per inch per deg. F.

Typical curves for deflections and stresses are presented in Figures 3 through 6 for three positions of the load. For example, in Figure 3a the top curve defines the deflection pattern when the moving load, represented by an arrow, is at the left edge of the slab. The middle and bottom curves define the deflection pattern when the load is 16 and 32 feet respectively from the left edge of the slab. In developing these curves, the following data was used.

$H = 8$ ins.

$K = 100, 200$ pci.

$C_r = 1.5, 2.0$

$v = 40$ mph.

$\Delta T = 30$ deg. F.

Figures 7 and 8 show how maximum deflection and maximum stress vary with velocity when the difference between slab surfaces is 20 and 40 degrees Fahrenheit. A record of the movement experienced by the ends

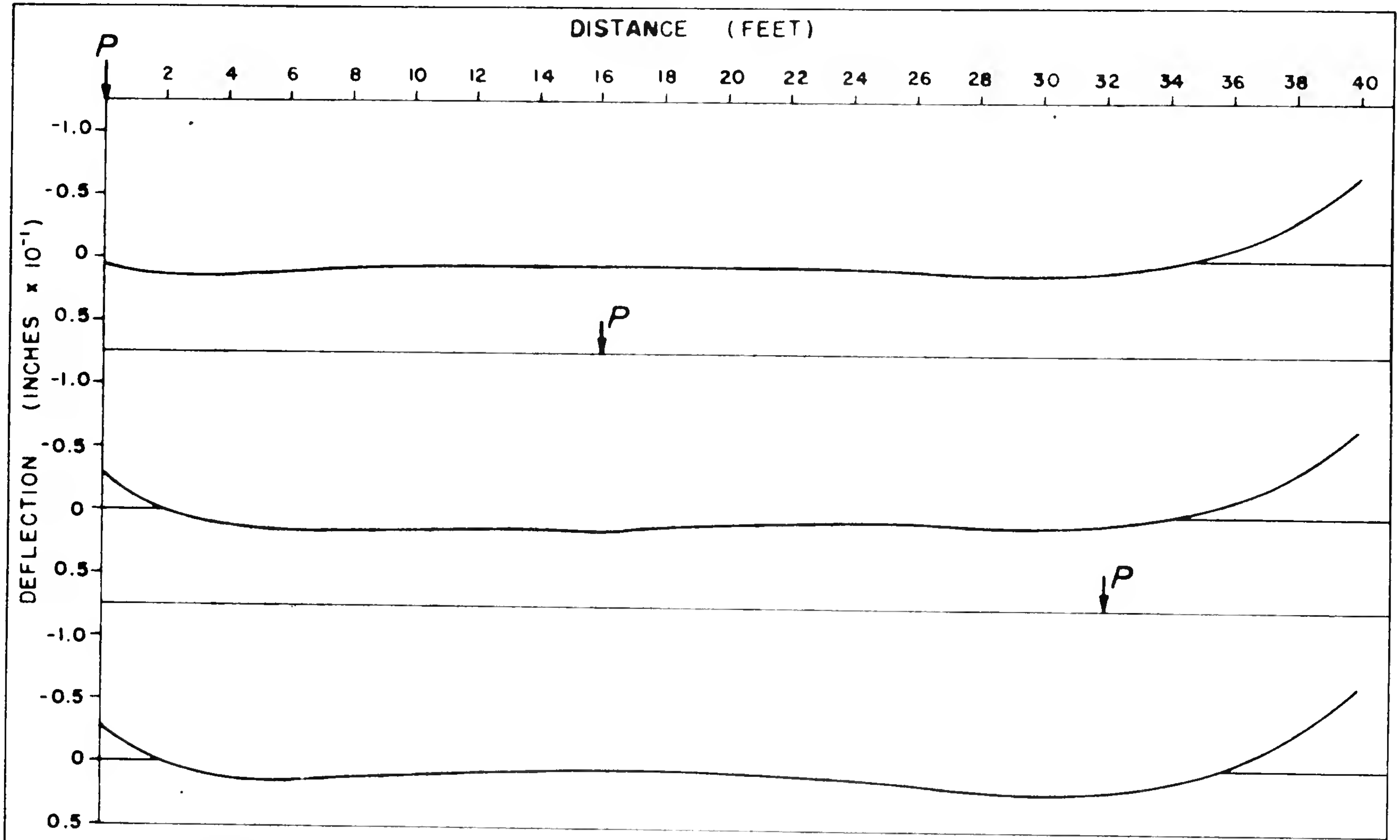



Figure 3a. Deflection versus Position of Moving Load ($K_o$ = 100 pci, H=8 ins. $C_r$=1.5, $\Delta T$=30 Deg., v=40 mph.)

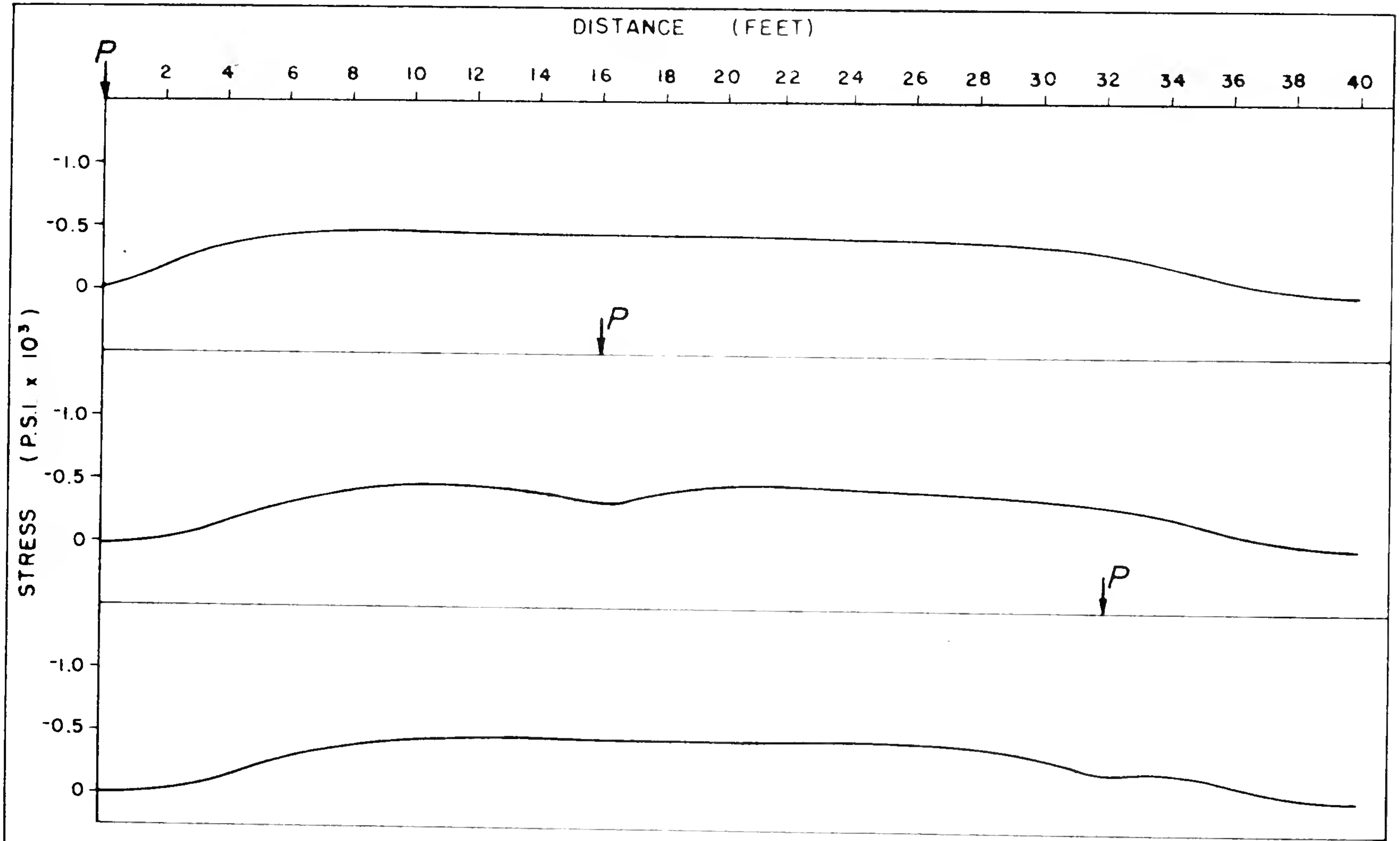



Figure 3b. Tensile Stress versus Position of Moving Load ($K_o$ = 100 pci., H=8 ins., $C_r$=1.5, $\Delta T$= 30 Deg., v=40 mph.)

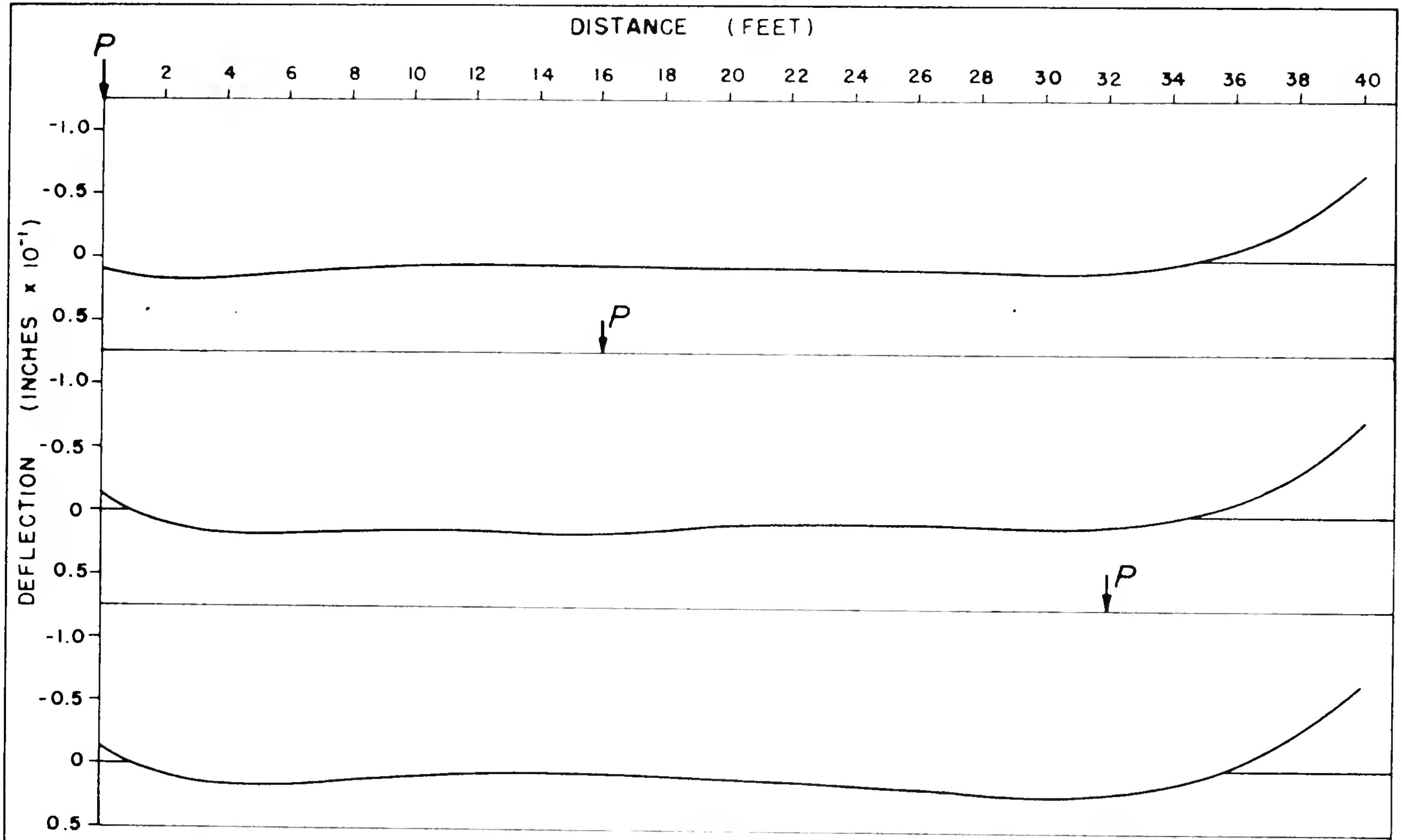



Figure 4a. Deflection versus Position of Moving Load ($K_o$ = 100 pci., H=8 ins., $C_r$=2.0, $\Delta T$=30 Deg., v=40 mph.)

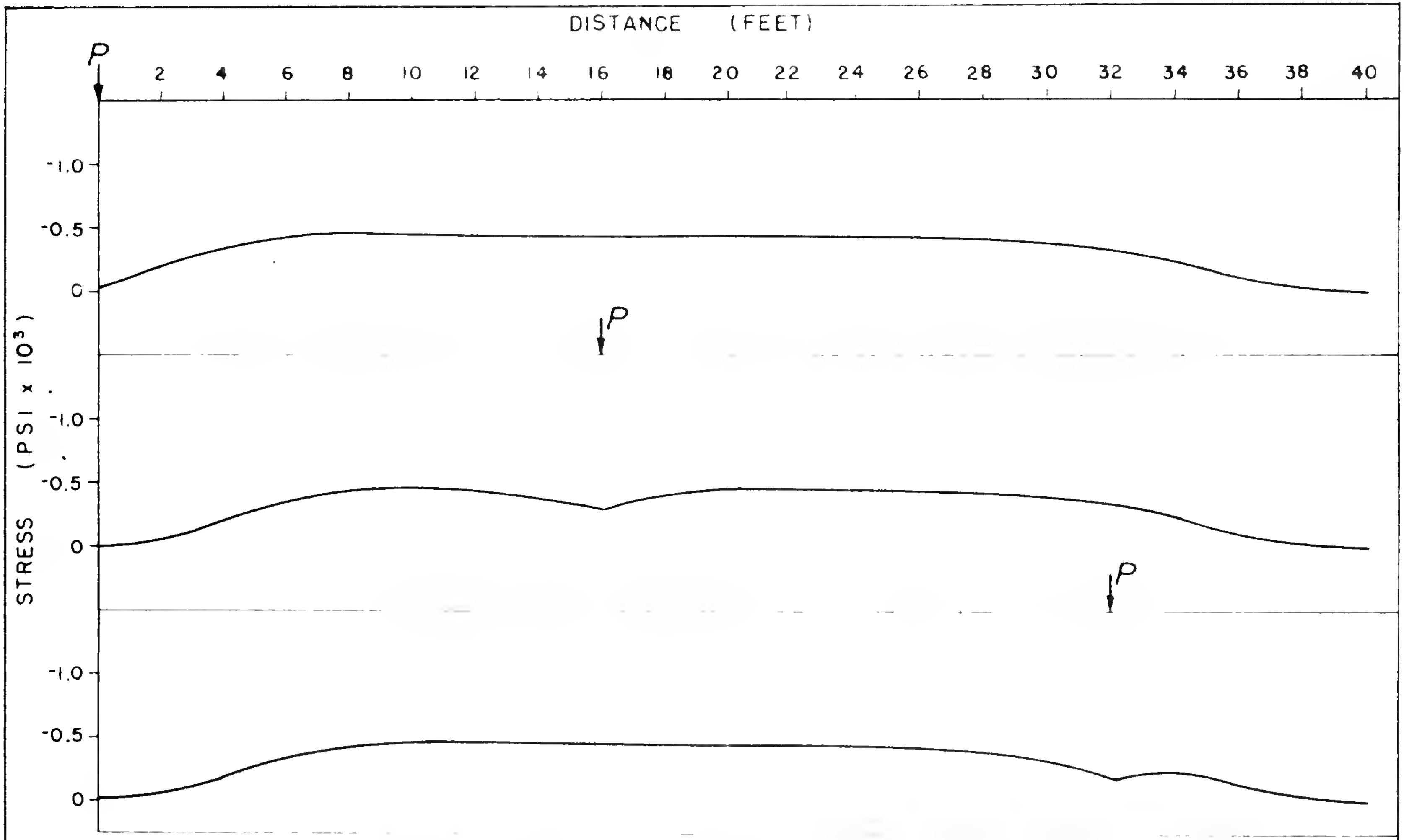



Figure 4b. Tensile Stress versus Position of Moving Load ($K_o$ = 100 pci., H=8 ins., $C_r$=2.0, $\Delta T$=30 Deg., v=40 mph.)

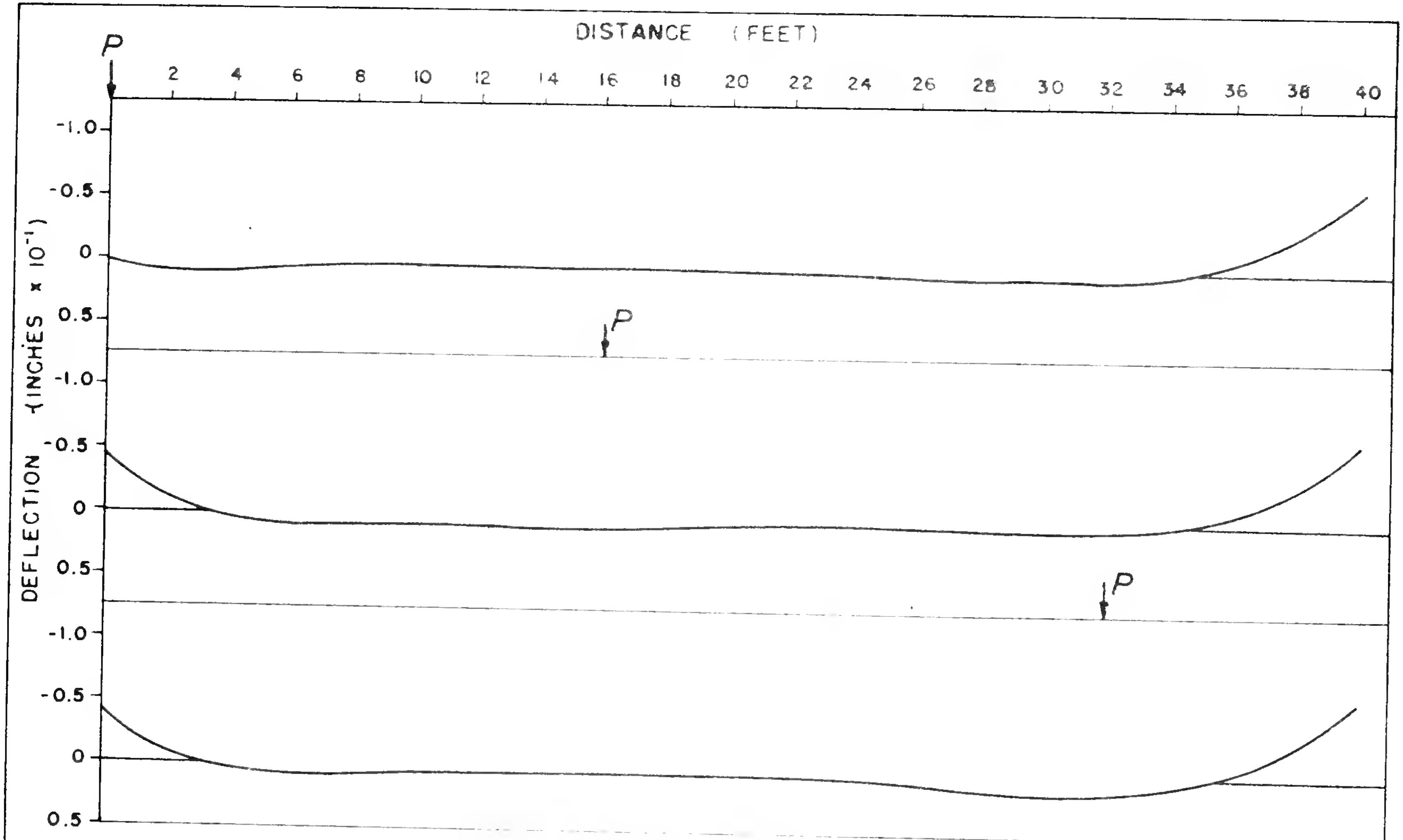



Figure 5a. Deflection versus Position of Moving Load ($K_o$ = 200 pci., H=8 ins., $C_r$=1.5, $\Delta T$=30 Deg. v=40 mph.)

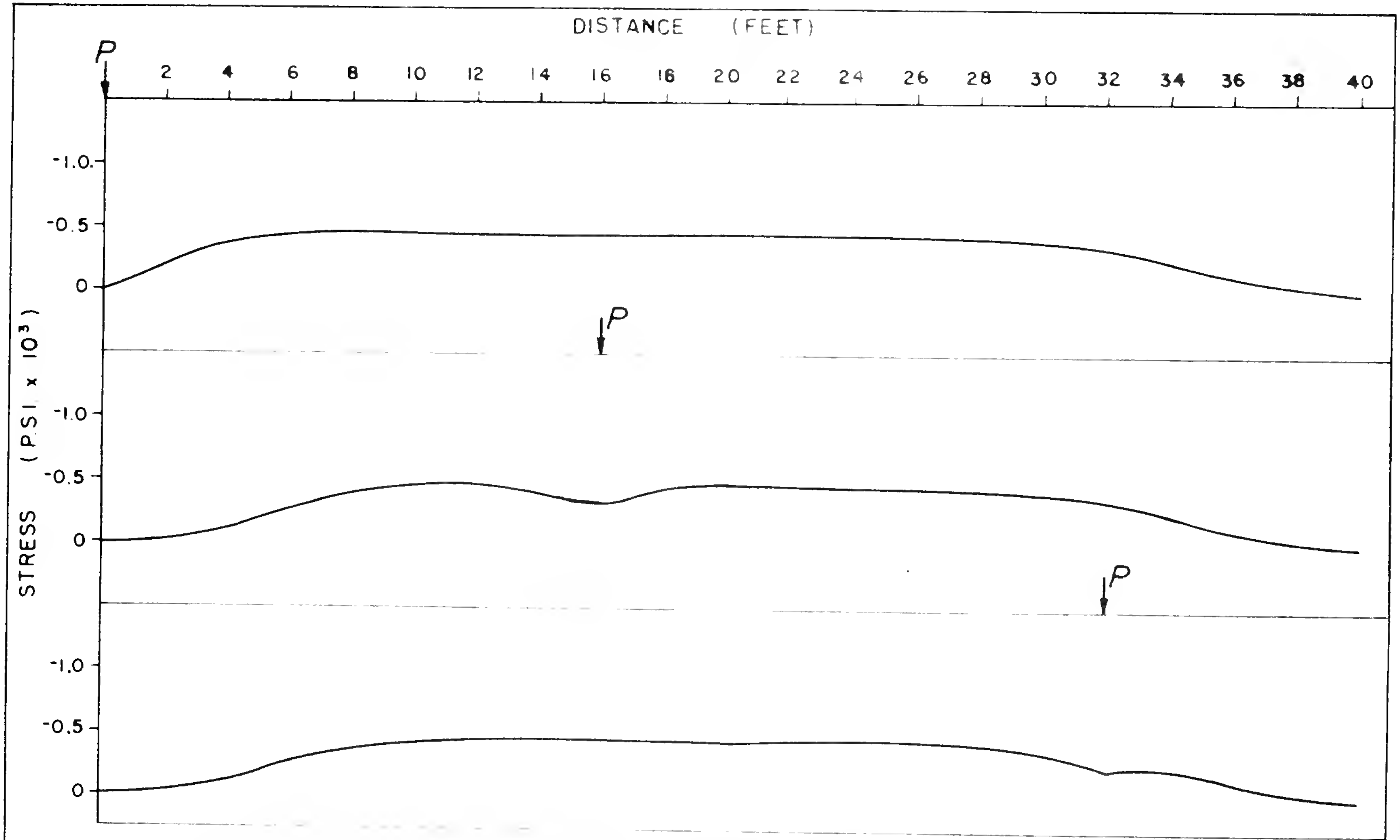



Figure 5b. Tensile Stress versus Position of Moving Load ($K_o$ = 200 pci., H=8 ins., $C_r$=1.5, $\Delta T$=30 Deg., v=40 mph.)

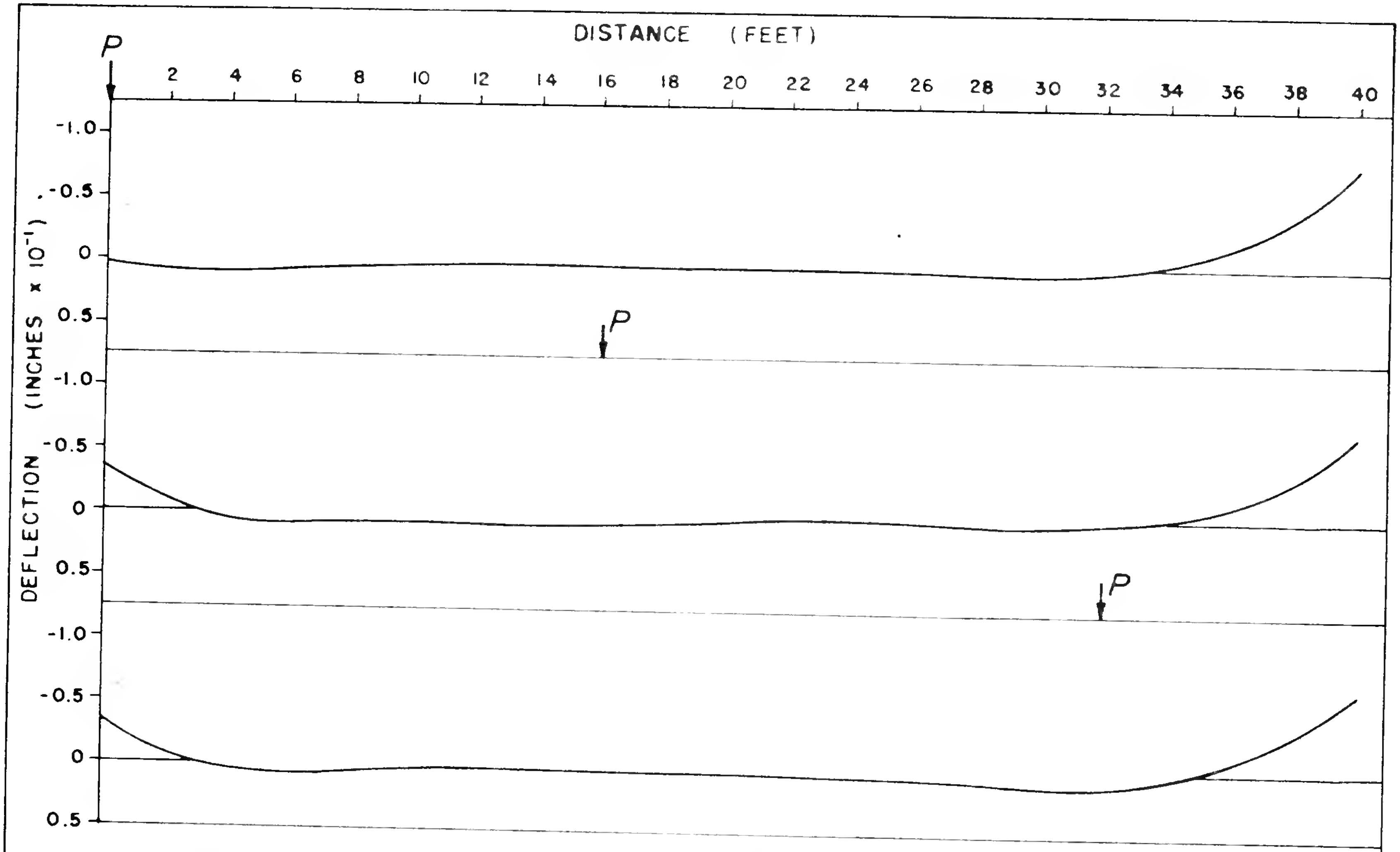



Figure 6a. Deflection versus Position of Moving Load ($K_o$ = 200 pci., H=8 ins., $C_r$=2.0, $\Delta T$=30 Deg., v=40 mph.)

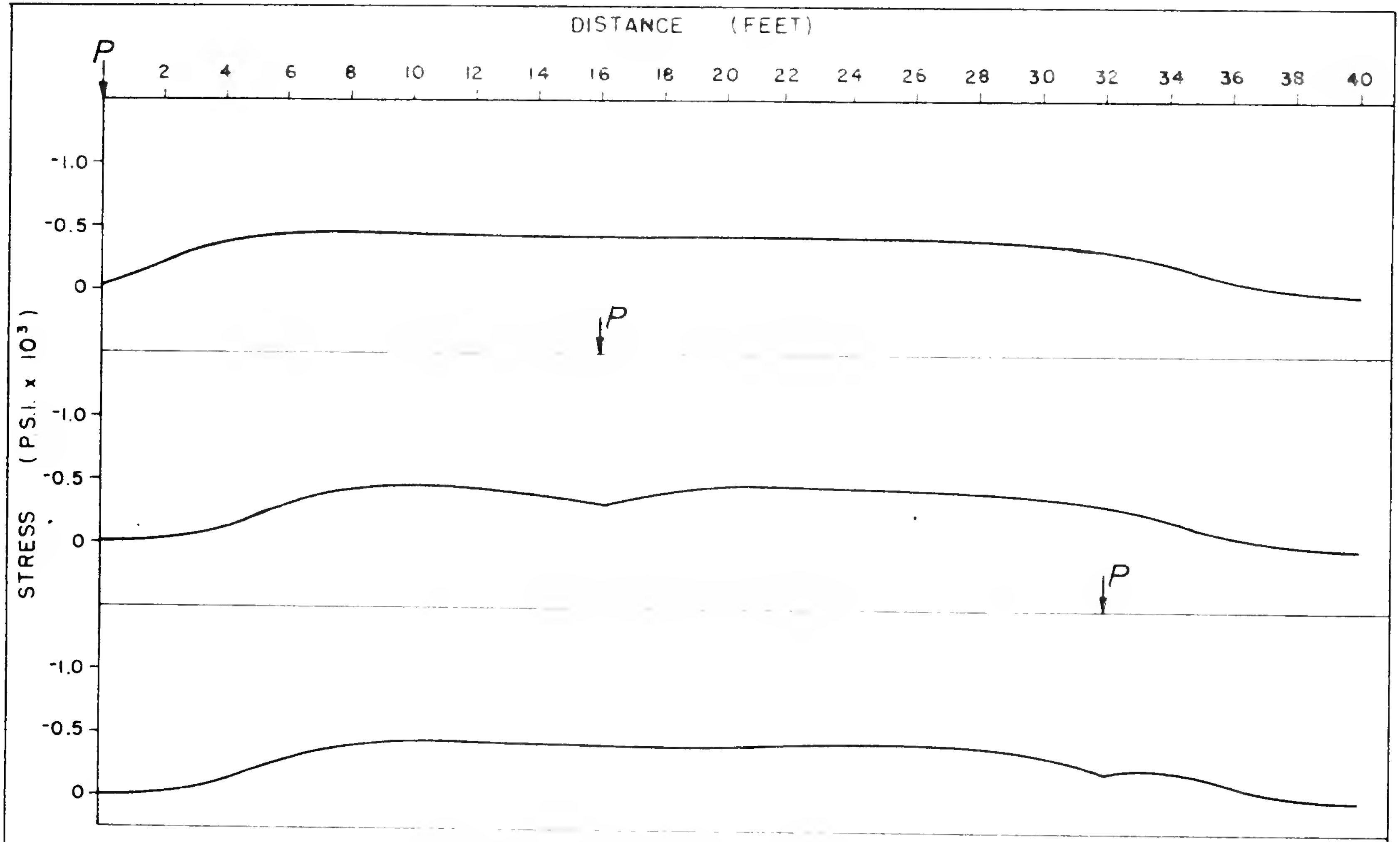



Figure 6b. Tensile Stress versus Position of Moving Load ($K_o$ = 200 pci., H=8 ins., $C_r$=2.0, $\Delta T$=30 Deg., v=40 mph.)

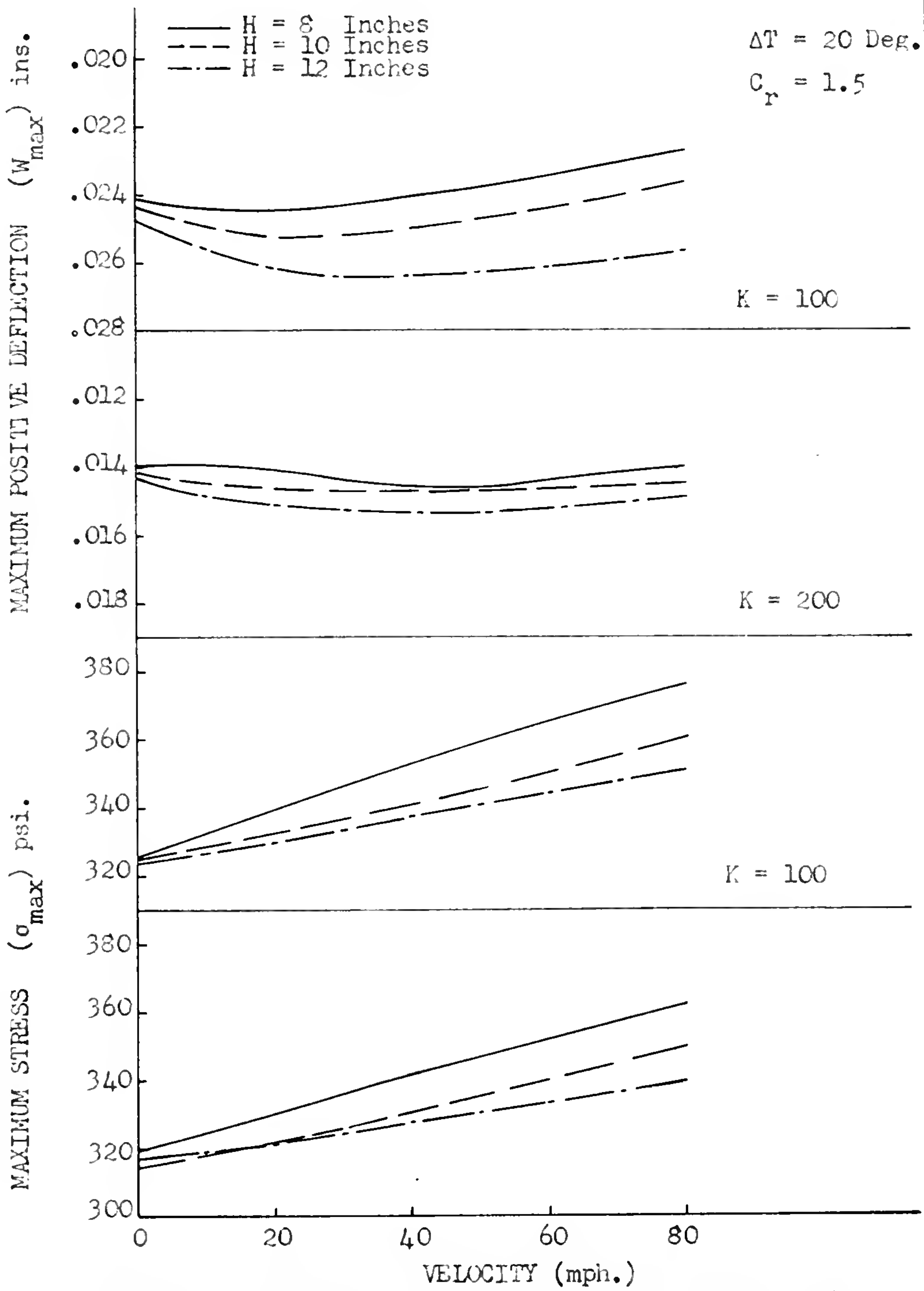



Figure 7a. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load ($C_r$=1.5, $\Delta T$=20 Deg.)

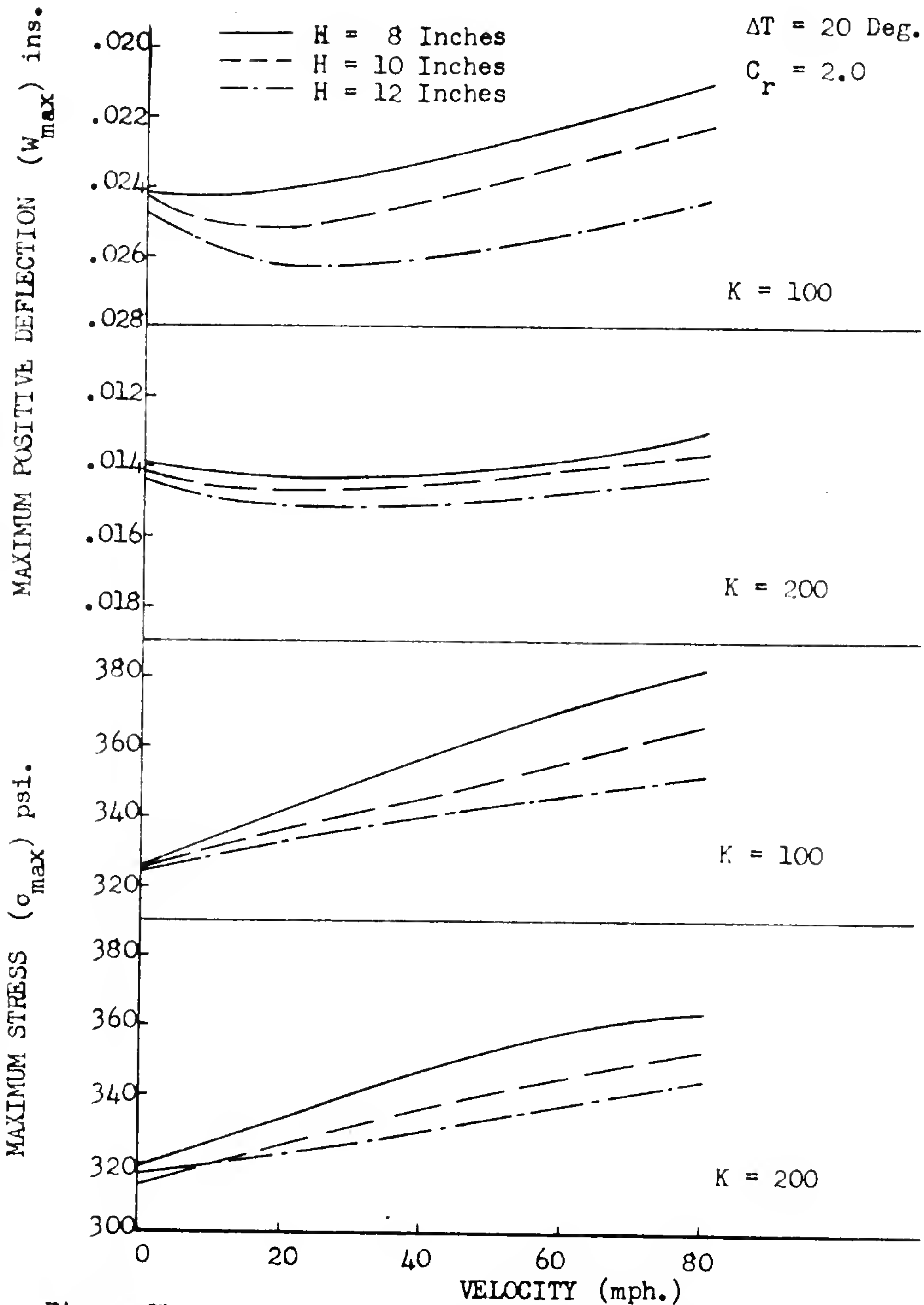



Figure 7b. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load ($C_r$=2.0, $\Delta T$=20 Deg.)

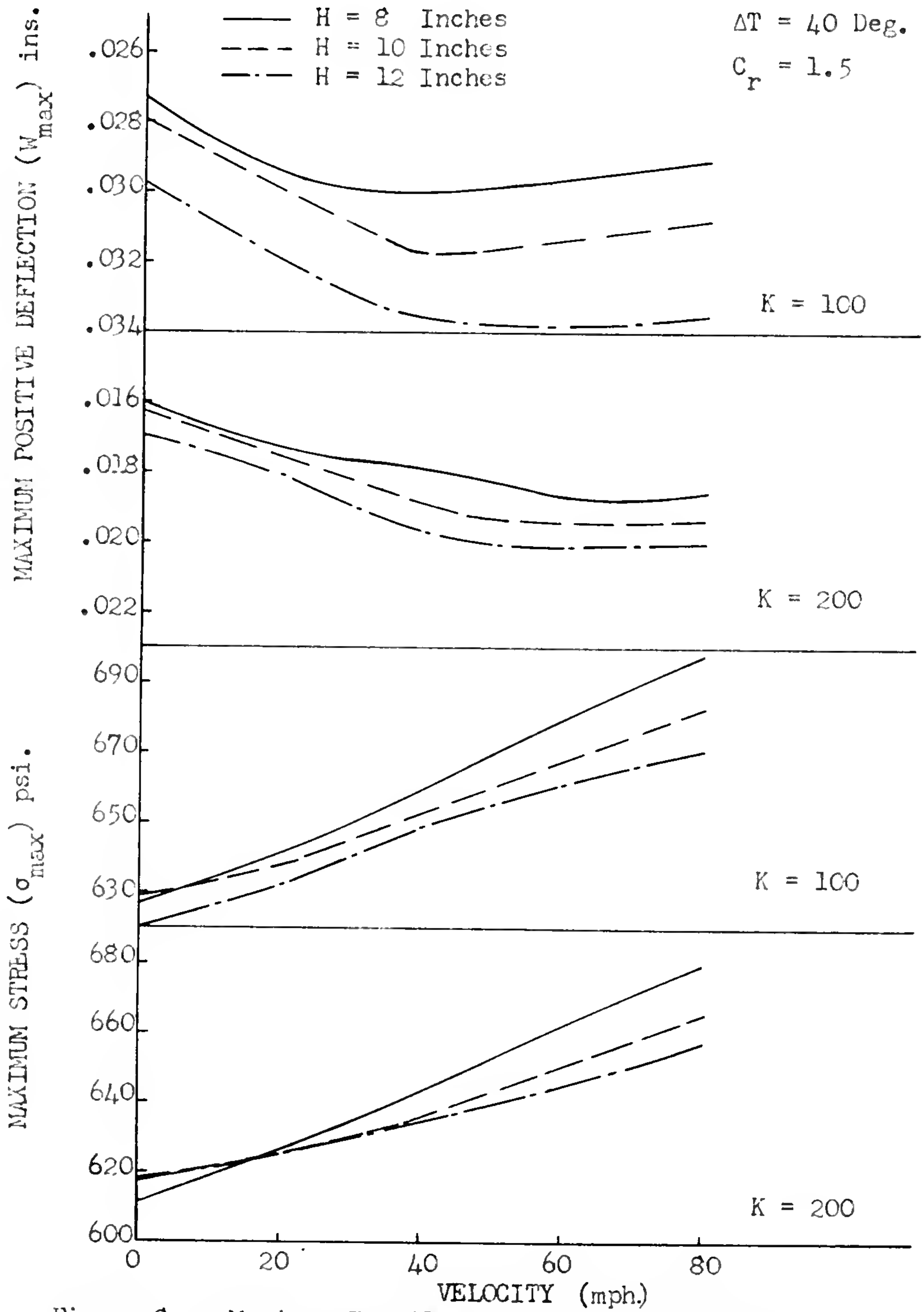



Figure 8a. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load ($C_r$=1.5, ΔT =40 Deg.)

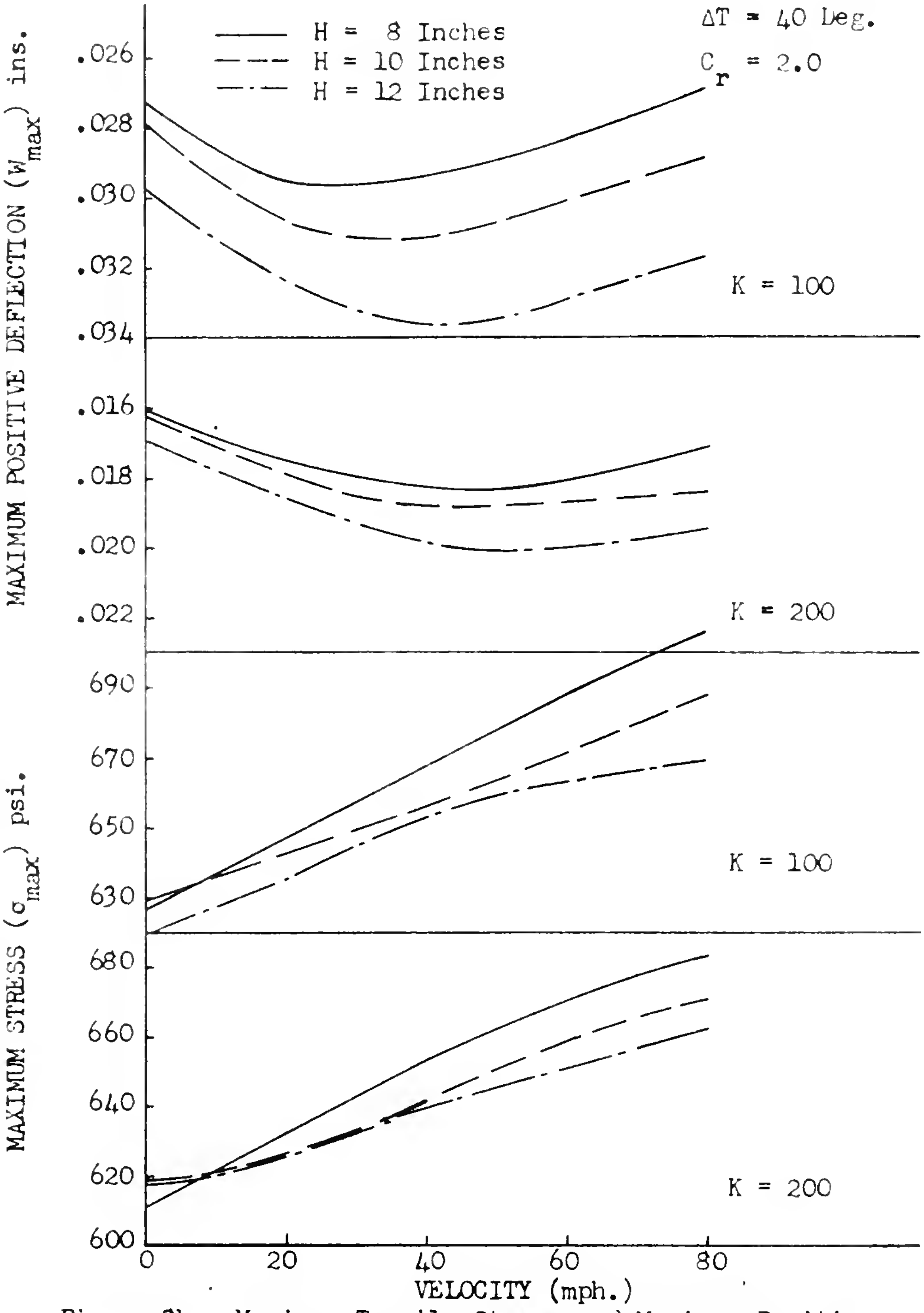



Figure 8b. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load ($C_r$=2.0, ΔT=40 Deg.)

of the slab is presented in Figures IV-1, IV-2 and IV-3 (see Appendix IV) as a function of the load's position.

## Discussion of Part I*

### Assumptions

In developing the theory for this part of the analysis, it was assumed that a pavement could be represented by an array of rectangular slabs. Furthermore, it was assumed that the bending moment between slabs could be neglected while a shearing force equivalent to that in an infinite slab could be used to provide shear transfer. This seemed justifiable because of the relatively short depth of dowel embedment and the general nature of expansion and contraction joints are such as to enable the transfer of only very limited bending. For most highway work, dowel bars are only approximately 2 feet long compared to the length of a slab which averages about 40 feet; moreover, they are, generally, smooth and lubricated at one end in order to maintain freedom of horizontal movement between slabs. Under repetitive loading, these joints become looser and conceivably act more like a hinge thus offering little or no moment transfer. On the other hand, substantial shear transfer could be experienced if the joint opening is small and the deflections are large enough to cause the development of a bearing pressure between dowel and concrete.

In any case, it should be noted that the magnitude of the moment and shear is really only significant in the near vicinity of the load

* The plan is to treat each part separately and then summarize them.

itself (see Figure IV-4). As a result, the solution is seen to be influenced by the values used for moment and shear transfer only when the load approaches the edges of the slab.

## Results

Figures 3 and 4 demonstrate that for the case studied here, the damping ratio, $C_r$, does not greatly influence the values obtained for deflection and stress at low to moderate velocities; however, the higher value of $C_r$ does result in a wider and less deflected trough.* In general, these figures tend to indicate that the pattern of the deflection and stress curves is mainly determined by temperature difference between slab surfaces and by the position of the moving load.

In the case considered here, temperature differences cause the ends of the slab to curl upwards and become unsupported, while points midway between the mid-point and the ends of the slab experience an increase in positive deflection. For the positions of load shown in Figures 3 through 6, there is an increase in positive deflection and a decrease in tensile stress in the near vicinity of the load; however, as Figures IV-1, IV-2, and IV-3, in Appendix IV indicate, the radius of influence (wave length of deflected surface) is not only a function of load position but also one of velocity. At low velocities, the radius of influence is small, but as the velocity increases, this radius increases significantly behind the moving load. As a matter of fact, this characteristic which was also observed by Thompson, plays an important role in the explanation of Figures 7 and 8.

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* The depression under the load is wider but of smaller amplitude.

For the case of an overdamped pavement ($C_r > 1.0$), it was initially anticipated that both the maximum deflection and stress would decrease with increasing velocity; however, Figures 7 and 8 indicate that if curling is in evidence this may not be the case. As was stated earlier, the maximum positive deflection in an unloaded slab subjected to moisture and/or temperature gradients which cause upward curling at the ends, occurs somewhere near the 1/4 and 3/4 points of the slab. When a moving load is introduced, a wave train is set up and the deflected trough which lags the load becomes wider as velocity increases. In other words, the influence of the deflected trough behind the moving load increases with increasing velocity.

Therefore, if a slab which is initially curled upwards at the ends is subjected to a moving load, there will be a tendency for the portion of the slab behind the load to become flatter and more fully supported as velocity increases, see Figures IV-1, IV-2, IV-3. As this flattening occurs, the maximum positive (downward) deflection behind the load increases until the velocity reaches a value which produces a reasonably flat, fully supported slab; then, a decrease in maximum positive deflection is experienced with further increases in velocity. Thus in a slab which is curled upwards at the ends the maximum positive deflection does not occur at velocity = 0, as is the case for fully supported slabs not subjected to moisture and/or temperature gradients, but at some velocity greater than zero.

As far as stresses are concerned, a line of reasoning similar to that used before to explain the deflection pattern, may be employed to account for the increase in maximum stress with increasing velocity as shown in Figures 7 and 8. Increased velocities result in a greater

tendency to flatten the slab and this in turn causes an increase in the tensile stress at the top of the slab.

As may be expected, the pattern as well as the magnitude of deflections and stresses obtained depend on the stiffness of the slab, the firmness of the subgrade material and the temperature difference existing between slab surfaces. For the range of velocity studied, maximum positive deflection decreased as the value of subgrade reaction increased, and increases were experienced as pavement thickness and temperature gradients became greater. However, as Figures 7 and 8 reveal, the greater the resistance to flattening, the higher is the velocity at which the curled surface becomes flat enough to result in a decrease in maximum positive deflection with increasing velocity. In the case of stresses, the increase in temperature gradient from 20 degrees to 40 degrees Fahrenheit produced almost a 50 percent increase in stress, while the variation in pavement thickness and subgrade reaction did not appear to have much influence. The influence of the higher value of $C_r$ on deflection and stress is also small, however the fact that it produces a wider and less deflected trough is quite evident.

# PART II

In this part of the thesis, the main objective is to determine what effect a reduction of the subgrade reaction would have on the stresses and deflections in a pavement subject to imposed vehicular loadings. For this case there is little loss in generality by assuming a particular value of shear and moment transfer between slabs. The procedure to be followed can be adapted to any degree of transfer, hence for the present purpose wherein there is primary concern only for the region evidencing subgrade reduction, it is expedient to consider primarily the case of an infinite slab. As a special case, the condition that no shear and moment transfer exist between two semi-infinite slabs is also examined.

The pavement section for the case of the infinite slab is illustrated in Figure 9a. Here, the subgrade in its original form is represented by zones 1 and 3, while the area over which the reduction of subgrade reaction occurs is represented by zone 2. The differential equation describing the surface of the pavement is given by Equation 2, in which $p(x_1, y_1, t) = C \frac{\partial w}{\partial t} + Kw$. Using the transformation $x = (x_1 - vt)$, the governing differential equation may be written as

$$D \frac{d^4 w}{dx^4} + \rho H v^2 \frac{d^2 w}{dx^2} - Cv \frac{dw}{dx} + Kw = q_o \qquad (29)$$

To introduce the equivalence of a moving load, another zone must be added to the pavement, for example in Figure 9b, zone 1 is seen to be

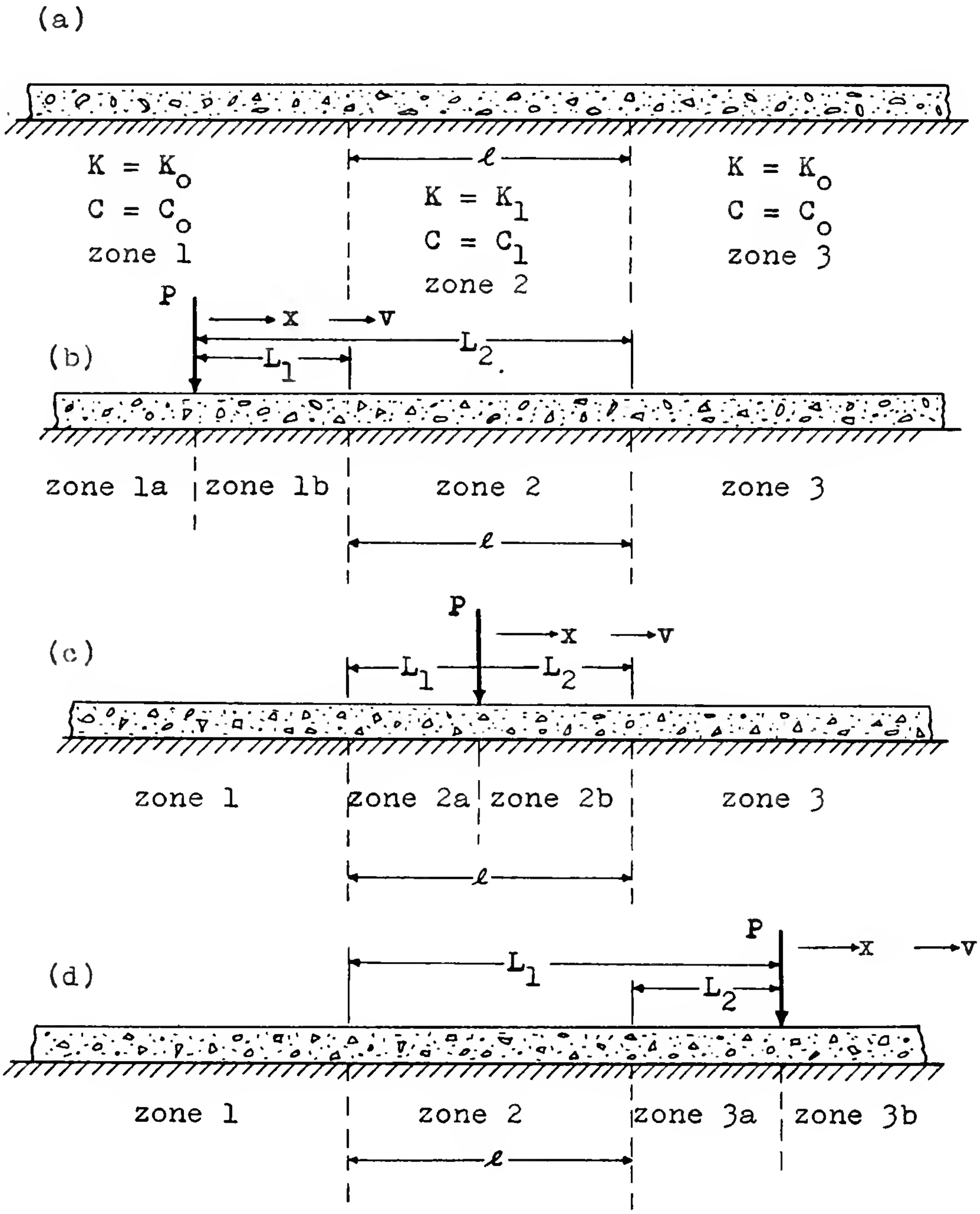


Figure 9. Infinite Slab Over Region of Reduced Subgrade Reaction

divided into two zones, 1a and 1b; and the force P is accounted for in the boundary conditions.

For the case when the moving load is over zone 1 and approaching zone 2, Equation 29 is applied to each of zones 1a, 1b, 2, and 3, and solutions are obtained using the appropriate subgrade properties. The boundary conditions used in evaluating the constants are as follows:

$$
\begin{aligned}
& w_{1a}(-\infty) = q_o/K_o \\
& w'_{1a}(-\infty) = 0 \\
& w_{1a}(0) = w_{1b}(0) \\
& w'_{1a}(0) = w'_{1b}(0) \\
& w''_{1a}(0) = w''_{1b}(0) \\
& w'''_{1b}(0) - w'''_{1a}(0) = P/D \\
& w_{1b}(L_1) = w_2(L_1) \\
& w'_{1b}(L_1) = w'_2(L_1) \\
& w''_{1b}(L_1) = w''_2(L_1) \\
& w'''_1(L_1) = w'''_2(L_1) \\
& w_2(L_2) = w_3(L_2) \\
& w'_2(L_2) = w'_3(L_2) \\
& w''_2(L_2) = w''_3(L_2) \\
& w'''_2(L_2) = w'''_3(L_2) \\
& w_3(\infty) = q_o/K_o \\
& w'_3(\infty) = 0
\end{aligned}
\qquad (30)
$$

## Solution of Problem

For the case where $\Delta_1$, $\Delta_2$ and $\Delta_3$ (see Equation 1) are all greater than zero and the loading is as illustrated in Figure 9b, the deflections and stresses in the slab are as follows:

For $x \leq 0$

$$w_{1a} = e^{a_1 x}(A_5 \sin b_1 x + A_6 \cos b_1 x) + q_o/K_o \tag{31}$$

$$\sigma_{1a} = \frac{-6De^{a_1 x}}{H^2}\Big[A_5\left[(a_1^2 - b_1^2)\sin b_1 x + 2a_1 b_1 \cos b_1 x\right] + A_6\left[(a_1^2 - b_1^2)\cos b_1 x - 2a_1 b_1 \sin b_1 x\right]\Big] \tag{32}$$

For $0 \leq x \leq L_1$

$$w_{1b} = e^{a_1 x}(B_5 \sin b_1 x + B_6 \cos b_1 x) + e^{-a_1 x}(B_7 \sin b_2 x + B_8 \cos b_2 x) + q_o/K_o \tag{33}$$

$$\begin{aligned}\sigma_{1b} = \frac{-6D}{H^2}\Big[&e^{a_1 x}\Big(B_5\left[(a_1^2 - b_1^2)\sin b_1 x + 2a_1 b_1 \cos b_1 x\right] + B_6\left[(a_1^2 - b_1^2)\cos b_1 x - 2a_1 b_1 \sin b_1 x\right]\Big) + \\ &e^{-a_1 x}\Big(B_7\left[(a_1^2 - b_2^2)\sin b_2 x - 2a_1 b_2 \cos b_2 x\right] + B_8\left[(a_1^2 - b_2^2)\cos b_2 x + 2a_1 b_2 \sin b_2 x\right]\Big)\Big]\end{aligned} \tag{34}$$

For $L_1 \leq x \leq L_2$

$$w_2 = e^{a_2 x}(C_5 \sin \gamma_1 x + C_6 \cos \gamma_1 x) + e^{-a_2 x}(C_7 \sin \gamma_2 x + C_8 \cos \gamma_2 x) +$$

$$q_o/K_1 \tag{35}$$

$$\sigma_2 = \frac{-6D}{H^2}\Big[e^{a_2 x}\Big(C_5[(a_2^2 - \gamma_1^2) \sin \gamma_1 x + 2a_2\gamma_1 \cos \gamma_1 x] +$$

$$C_6[(a_2^2 - \gamma_1^2) \cos \gamma_1 x - 2a_2\gamma_1 \sin \gamma_1 x]\Big) +$$

$$e^{-a_2 x}\Big(C_7[(a_2^2 - \gamma_2^2) \sin \gamma_2 x - 2a_2\gamma_2 \cos \gamma_2 x] +$$

$$C_8[2a_2\gamma_2 \sin \gamma_2 x + (a_2^2 - \gamma_2^2) \cos \gamma_2 x]\Big)\Big] \tag{36}$$

For $x \geq L_2$

$$w_3 = e^{-a_1 x}(D_7 \sin b_2 x + D_8 \cos b_2 x) + q_o/K_o \tag{37}$$

$$\sigma_3 = \frac{-6D}{H^2} e^{-a_1 x}\Big[D_7[(a_1^2 - b_2^2) \sin b_2 x - 2a_1 b_2 \cos b_2 x] +$$

$$D_8[(a_1^2 - b_2^2) \cos b_2 x + 2a_1 b_2 \sin b_2 x]\Big] \tag{38}$$

Applying the conditions listed in Equation 30 to Equations 31, 33, 35 and 37, the following set of linear equations are obtained.

$$A_6 - B_6 - B_8 = 0 \tag{39}$$

$$b_1A_5 + a_1A_6 - b_1B_5 - a_1B_6 - b_2B_7 + a_1B_8 = 0 \tag{40}$$

$$2a_1b_1A_5 + (a_1^2 - b_1^2)\, A_6 - 2a_1b_1B_5 - (a_1^2 - b_1^2)\, B_6 -$$
$$(a_1^2 - b_2^2)\, B_8 + 2a_1b_2B_7 = 0 \tag{41}$$

$$(b_1^3 - 3b_1a_1^2)\, A_5 - (a_1^3 - 3a_1b_1^2)\, A_6 - (b_1^3 - 3b_1a_1^2)\, B_5 +$$
$$(a_1^3 - 3a_1b_1^2)\, B_6 + (3a_1^2b_2 - b_2^3)\, B_7 + (3a_1b_2^3 - a_1^3)\, B_8 = P/D \tag{42}$$

$$e^{a_1L_1}(B_5 \sin b_1L_1 + B_6 \cos b_1L_1) + e^{-a_1L_1}(B_7 \sin b_2L_1 + B_8 \cos b_2L_1) -$$
$$e^{a_2L_1}(C_5 \sin \gamma_1L_1 + C_6 \cos \gamma_1L_1) - e^{-a_2L_1}(C_7 \sin \gamma_2L_1 + C_8 \cos \gamma_2L_1)$$
$$= q_o\left(\frac{K_o - K_1}{K_1\, K_o}\right) \tag{43}$$

$$e^{a_1L_1}\left[ B_5(b_1 \cos b_1L_1 + a_1 \sin b_1L_1) + B_6(a_1 \cos b_1L_1\right.$$
$$\left. - b_1 \sin b_1L_1) \right] + e^{-a_1L_1}\left[B_7(b_2 \cos b_2L_1 - a_1 \sin b_2L_1)\right.$$
$$\left. - B_8(b_2 \sin b_2L_1 + a_1 \cos b_2L_1)\right] - e^{a_2L_1}\left[C_5(\gamma_1 \cos \gamma_1L_1\right.$$
$$\left. + a_2 \sin \gamma_1L_1) + C_6(a_2 \cos \gamma_1L_1 - \gamma_1 \sin \gamma_1L_1) \right] + e^{-a_2L_1}\left[C_8(\gamma_2 \sin \gamma_2L_1\right.$$
$$\left. + a_2 \cos \gamma_2L_1) - C_7(\gamma_2 \cos \gamma_2L_1 - a_2 \sin \gamma_2L_1)\right] = 0 \tag{44}$$

$$e^{a_1L_1}\Big[B_5\left[(a_1^2 - b_1^2)\sin b_1L_1 + 2a_1b_1\cos b_1L_1\right]+$$

$$B_6\left[(a_1^2 - b_1^2)\cos b_1L_1 - 2a_1b_1\sin b_1L_1\right]\Big]+$$

$$e^{-a_1L_1}\Big[B_7\left[(a_1^2 - b_2^2)\sin b_2L_1 - 2a_1b_2\cos b_2L_1\right]+$$

$$B_8\left[(a_1^2 - b_2^2)\cos b_2L_1 + 2a_1b_2\sin b_2L_1\right]\Big]$$

$$- e^{a_2L_1}\Big[C_5\left[(a_2^2 - \gamma_1^2)\sin \gamma_1L_1 + 2a_2\gamma_1\cos \gamma_1L_1\right]+$$

$$C_6\left[(a_2^2 - \gamma_1^2)\cos \gamma_1L_1 - 2a_2\gamma_1\sin \gamma_1L_1\right]\Big]-$$

$$e^{-a_2L_1}\Big[C_7\left[(a_2^2 - \gamma_2^2)\sin \gamma_2L_1 - 2a_2\gamma_2\cos \gamma_2L_1\right]+$$

$$C_8\left[2a_2\gamma_2\sin \gamma_2L_1 + (a_2^2 - \gamma_2^2)\cos \gamma_2L_1\right]\Big] = 0 \qquad (45)$$

$$e^{a_1L_1}\Big[B_5[(a_1^3 - 3a_1b_1^2)\sin b_1L_1 - (b_1^3 - 3b_1a_1^2)\cos b_1L_1] +$$

$$B_6[(b_1^3 - 3b_1a_1^2)\sin b_1L_1 + (a_1^3 - 3a_1b_1^2)\cos b_1L_1]\Big] +$$

$$e^{-a_1L_1}\Big[B_7[(3a_1^2b_2 - b_2^3)\cos b_2L_1 + (3a_1b_2^2 - a_1^3)\sin b_2L_1] +$$

$$B_8[(3a_1b_2^2 - a_1^3)\cos b_2L_1 + (b_2^3 - 3a_1^2b_2)\sin b_2L_1]\Big] -$$

$$e^{a_2L_1}\Big[C_5[(a_2^3 - 3a_2\gamma_1^2)\sin \gamma_1L_1 - (\gamma_1^3 - 3\gamma_1a_2^2)\cos \gamma_1L_1] +$$

$$C_6[(\gamma_1^3 - 3\gamma_1a_2^2)\sin \gamma_1L_1 + (a_2^3 - 3a_2\gamma_1^2)\cos \gamma_1L_1]\Big] -$$

$$e^{-a_2L_1}\Big[C_7[(3a_2^2\gamma_2 - \gamma_2^3)\cos \gamma_2L_1 + (3a_2\gamma_2^2 - a_2^3)\sin \gamma_2L_1] +$$

$$C_8[(3a_2\gamma_2^2 - a_2^3)\cos \gamma_2L_1 + (\gamma_2^3 - 3a_2^2\gamma_2)\sin \gamma_2L_1]\Big] = 0 \qquad (46)$$

$$e^{a_2L_2}(C_5 \sin \gamma_1L_2 + C_6 \cos \gamma_1L_2) + e^{-a_2L_2}(C_7 \sin \gamma_2L_2 + C_8 \cos \gamma_2L_2)$$

$$- e^{-a_1L_2}(D_7 \sin b_2L_2 + D_8 \cos b_2L_2) = q_o\left(\frac{K_1 - K_o}{K_1 K_o}\right) \qquad (47)$$

$$e^{a_2 L_2}[C_5 (\gamma_1 \cos \gamma_1 L_2 + a_2 \sin \gamma_1 L_2) + C_6 (a_2 \cos \gamma_1 L_2 -$$

$$\gamma_1 \sin \gamma_1 L_2)] + e^{-a_2 L_2}[C_7 (\gamma_2 \cos \gamma_2 L_2 - a_2 \sin \gamma_2 L_2) -$$

$$C_8 (\gamma_2 \sin \gamma_2 L_2 + a_2 \cos \gamma_2 L_2)] \quad -e^{-a_1 L_2} [D_7 (b_2 \cos b_2 L_2 -$$

$$a_1 \sin b_2 L_2) - D_8 (b_2 \sin b_2 L_2 + a_1 \cos b_2 L_2)] = 0 \qquad (48)$$

$$e^{a_2 L_2} \Big[ C_5 [(a_2^2 - \gamma_1^2) \sin \gamma_1 L_2 + 2a_2 \gamma_1 \cos \gamma_1 L_2 ] +$$

$$C_6 [(a_2^2 - \gamma_1^2) \cos \gamma_1 L_2 - 2a_2 \gamma_1 \sin \gamma_1 L_2 ] \Big] + e^{-a_2 L_2} \Big[ C_7 [(a_2^2 - \gamma_2^2) \sin \gamma_2 L_2$$

$$- 2a_2 \gamma_2 \cos \gamma_2 L_2] + C_8 [2a_2 \gamma_2 \sin \gamma_2 L_2 + (a_2^2 - \gamma_2^2) \cos \gamma_2 L_2] \Big] -$$

$$e^{-a_1 L_2} \Big[ D_7 [(a_1^2 - b_2^2) \sin b_2 L_2 - 2a_1 b_2 \cos b_2 L_2] +$$

$$D_8 [2a_1 b_2 \sin b_2 L_2 + (a_1^2 - b_2^2) \cos b_2 L_2 ] \Big] = 0 \qquad (49)$$

$$e^{a_2L_2}\Big[C_5\,[(a_2^3 - 3a_2\gamma_1^2)\sin\gamma_1L_2 - (\gamma_1^3 - 3\gamma_1a_2^2)\cos\gamma_1L_2]+$$

$$C_6[(\gamma_1^3 - 3\gamma_1a_2^2)\sin\gamma_1L_2 + (a_2^3 - 3a_2\gamma_1^2)\cos\gamma_1L_2]\Big] +$$

$$e^{-a_2L_2}\Big[C_7\,[(3a_2^2\gamma_2 - \gamma_2^3)\cos\gamma_2L_2 + (3a_2\gamma_2^2 - a_2^3)\sin\gamma_2L_2]+$$

$$C_8[(3a_2\gamma_2^2 - a_2^3)\cos\gamma_2L_2 + (\gamma_2^3 - 3a_2^2\gamma_2)\sin\gamma_2L_2]\Big] -$$

$$e^{-a_1L_2}\Big[D_7\,[(3a_1^2b_2 - b_2^3)\cos b_2L_2 + (3a_1b_2^2 - a_1^3)\sin b_2L_2]+$$

$$D_8[(3a_1b_2^2 - a_1^3)\cos b_2L_2 + (b_2^3 - 3a_1^2b_2)\sin b_2L_2]\Big] = 0 \qquad (50)$$

The constants in Equations 39 through 50 were evaluated using the Method of Crout Reduction (25) with an IBM 7090 computer. In addition, a column pivot search was performed during reduction in order to improve accuracy. After evaluating the constants, the deflections and stresses were determined. The complete solution to the problem was obtained by applying this same procedure with suitable modifications to the schemes shown in Figure 9c and 9d. The general solution for the case where $\Delta_1$ and $\Delta_3 > 0$ but $\Delta_2 < 0$ is given in Appendix II.

To examine the effects of a loss in load transfer at a joint or crack assumed to exist at the mid-point of the weakened zone, the special case illustrated in Figure 10 was investigated. Here, the moment and shear acting at points e and f, are assumed to be respectively 0% and 50% of that which would exist in a corresponding section of an infinite slab. Using the same procedure detailed above, a set of linear equations

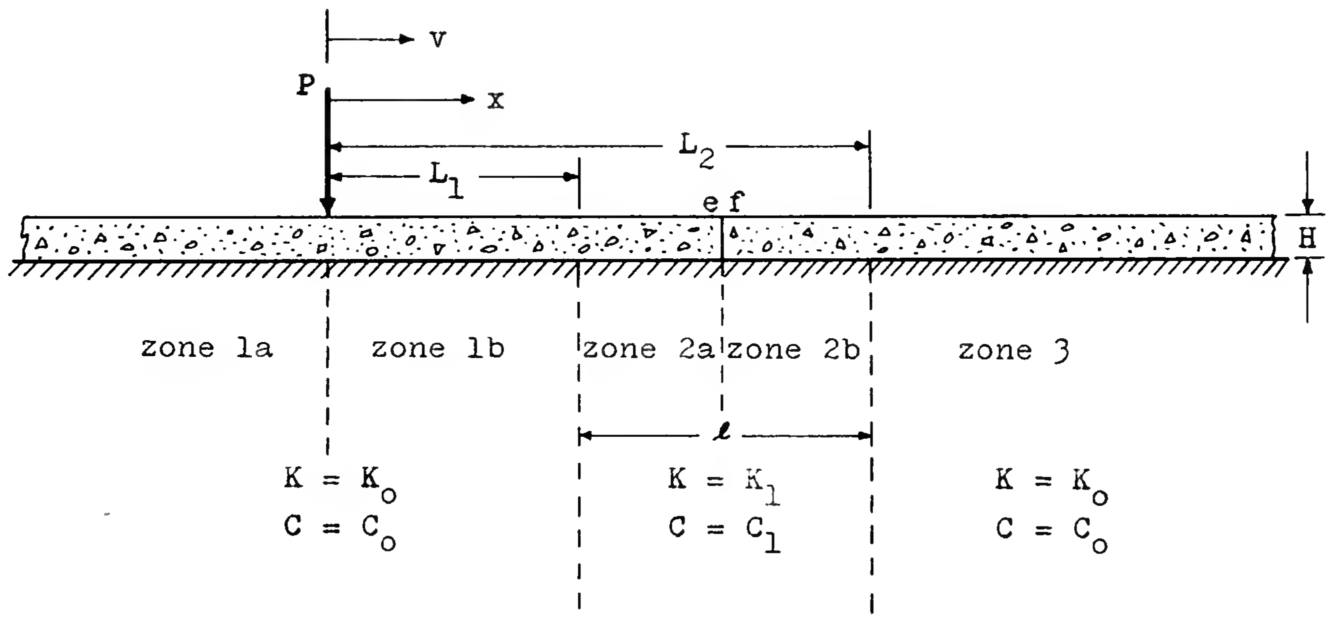


Figure 10. Section of Infinite Slab over Region of Reduced Subgrade Reaction

were developed. The constants were evaluated and the stresses and deflections determined.

## Results

To investigate the effect of a reduction in subgrade reaction, numerical results were obtained for combinations of the following data:

$\mu = 0.15$

$E = 4 \times 10^6$ psi

$\ell = 24, 48, 96$ ins.

$H = 8, 10, 12$ ins.

$K_o = 100, 200$ pci

$K_1 = 0, 25, 50, 75, 100, 150, 200$ pci

$C_r = 1.5, 2.0$

$v = 0, 20, 40, 60, 80$ mph

$P = 125$ lbs/in.

$q_o = 150.9$ pcf

Some typical curves (in non-dimensional form) for displacements and stresses for three positions of the load are presented in Figures 11, 12 and 13.

In Figures 11 and 13, the moving load (represented by the vector, P) is 2 feet away from the nearest point of the zone of reduced subgrade reaction, $K_1$, while in Figure 12, the load is at the mid-point of this zone.

Values for the deflection and stress under a static load, P, are given in Table 1. These values may be used in Figures 11 to 13 with the appropriate amplification ratios, to obtain the magnitudes of the deflection

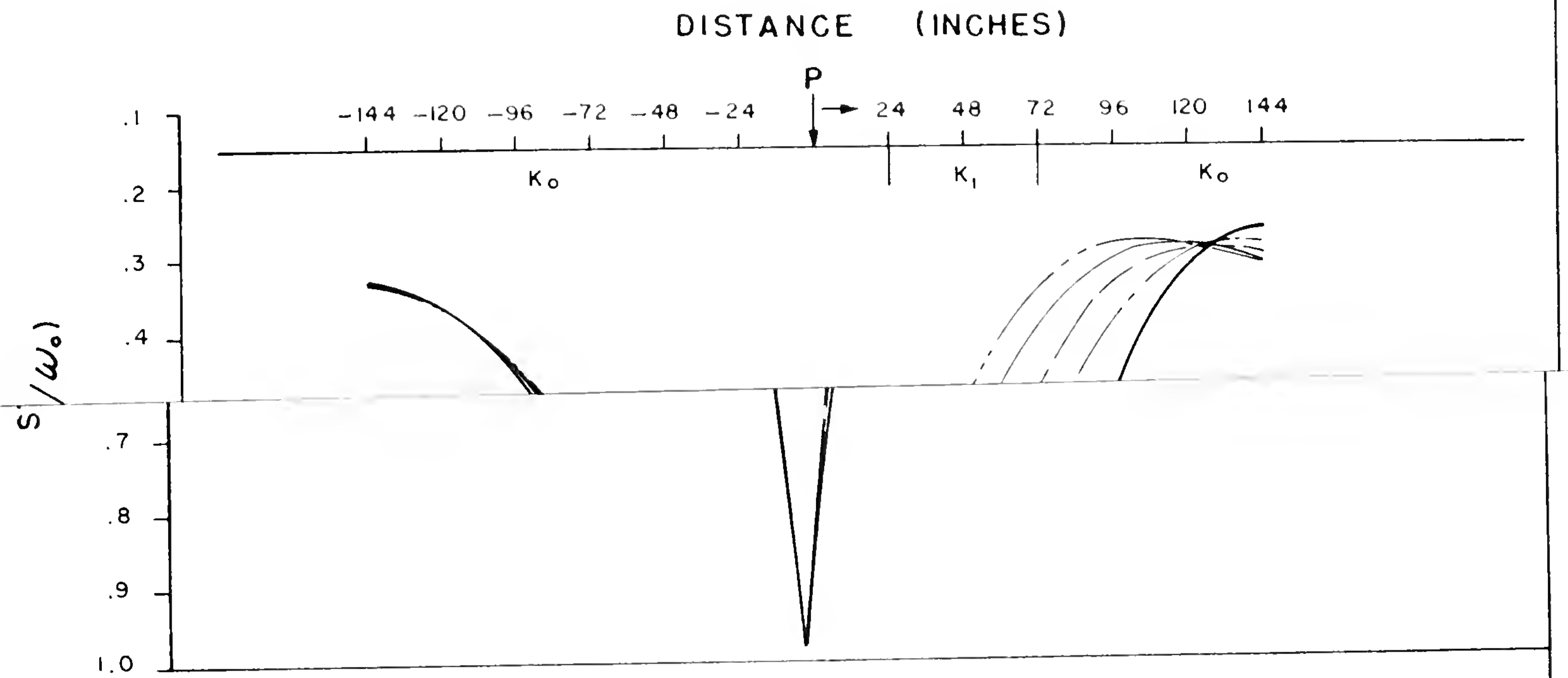


Figure 11. Deflection and Stress Amplification for Load 2 Ft. Behind Region of Reduced Subgrade Reaction

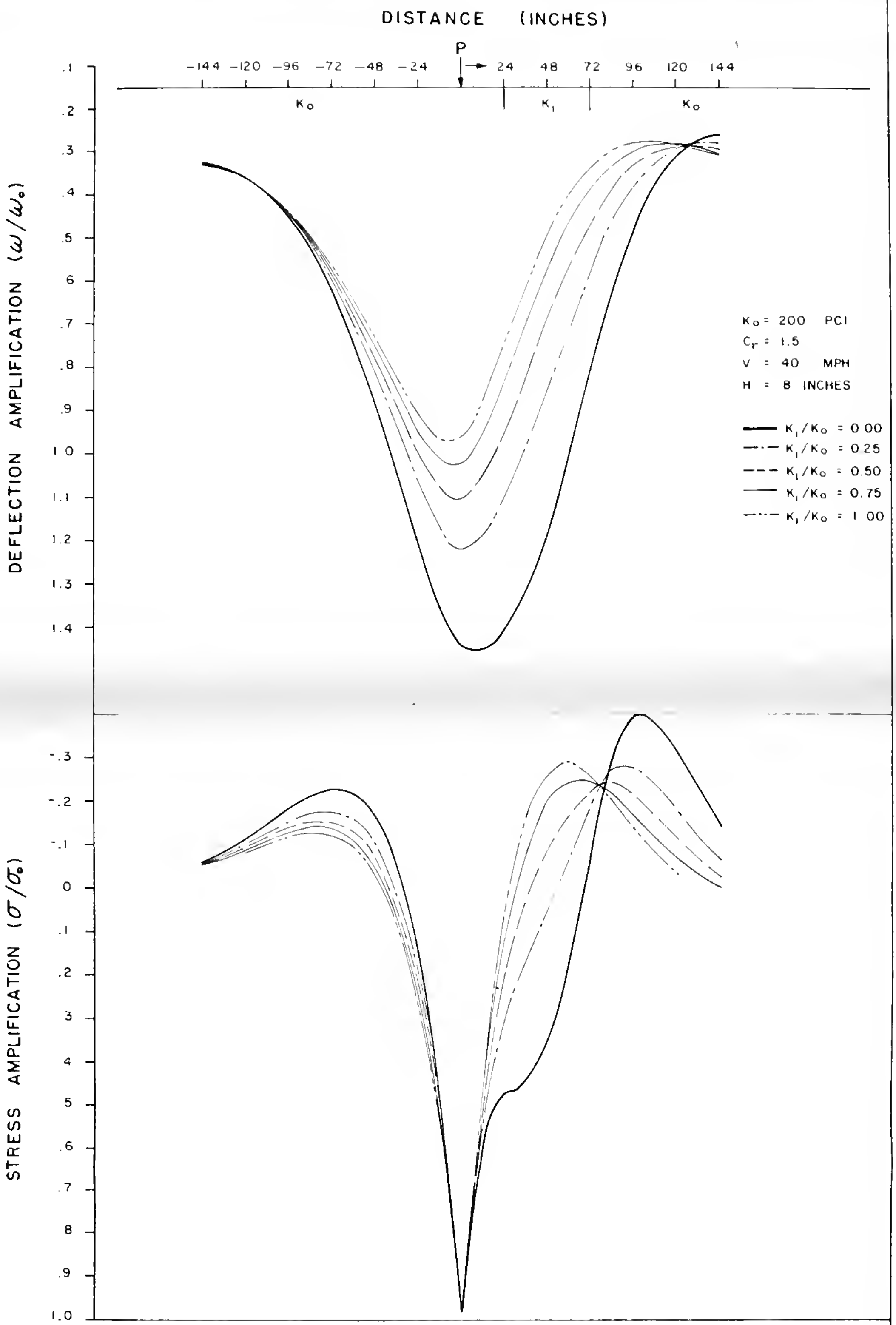


Figure 11. Deflection and Stress Amplification for Load 2 Ft. Behind Region of Reduced Subgrade Reaction

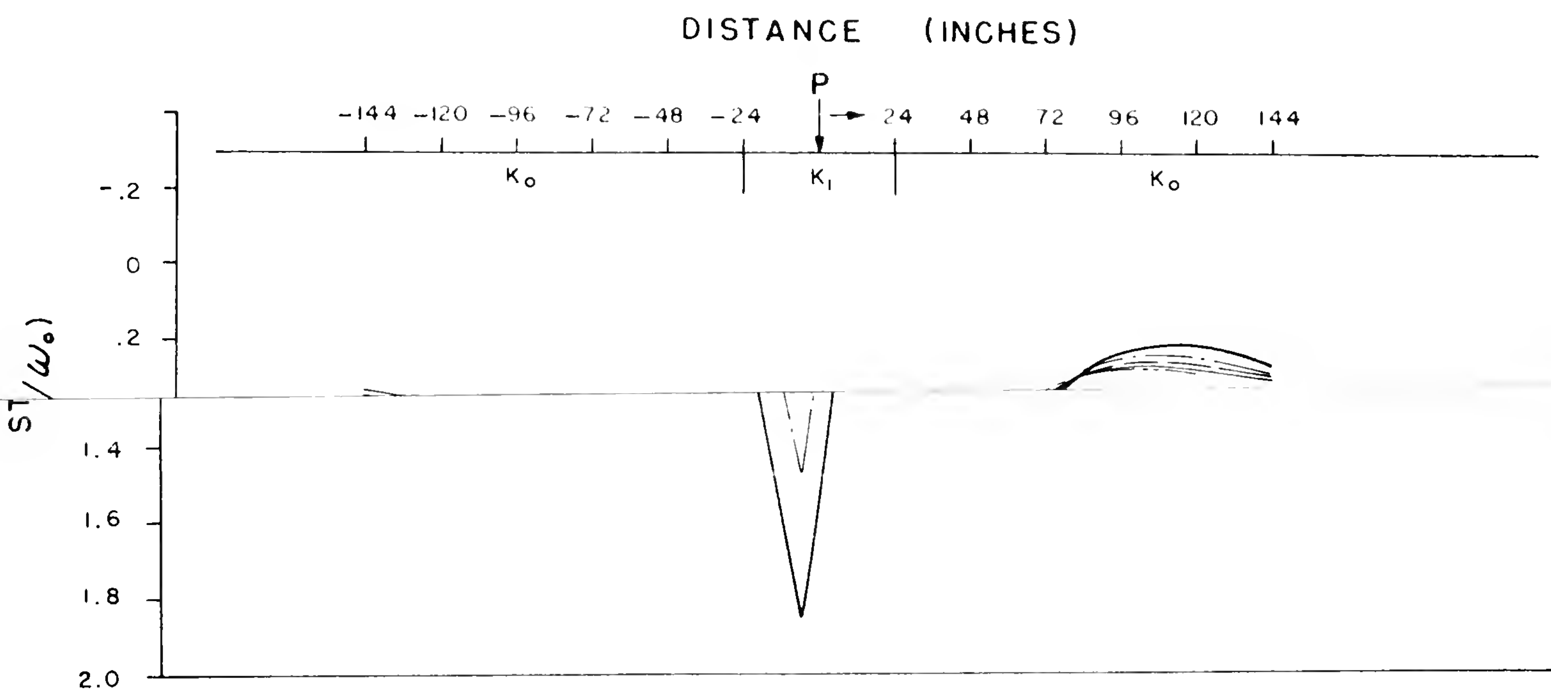


Figure 12. Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction

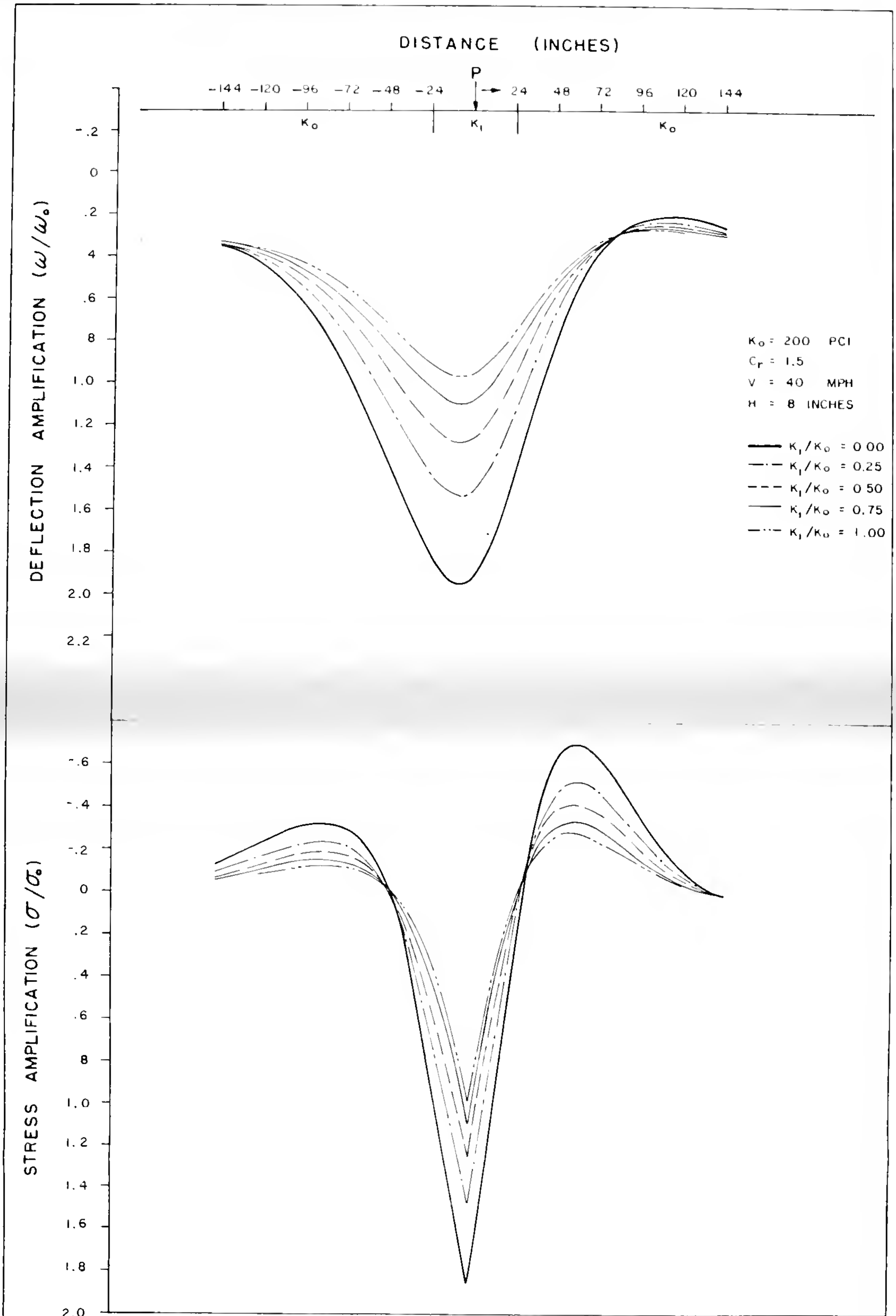


Figure 12. Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction

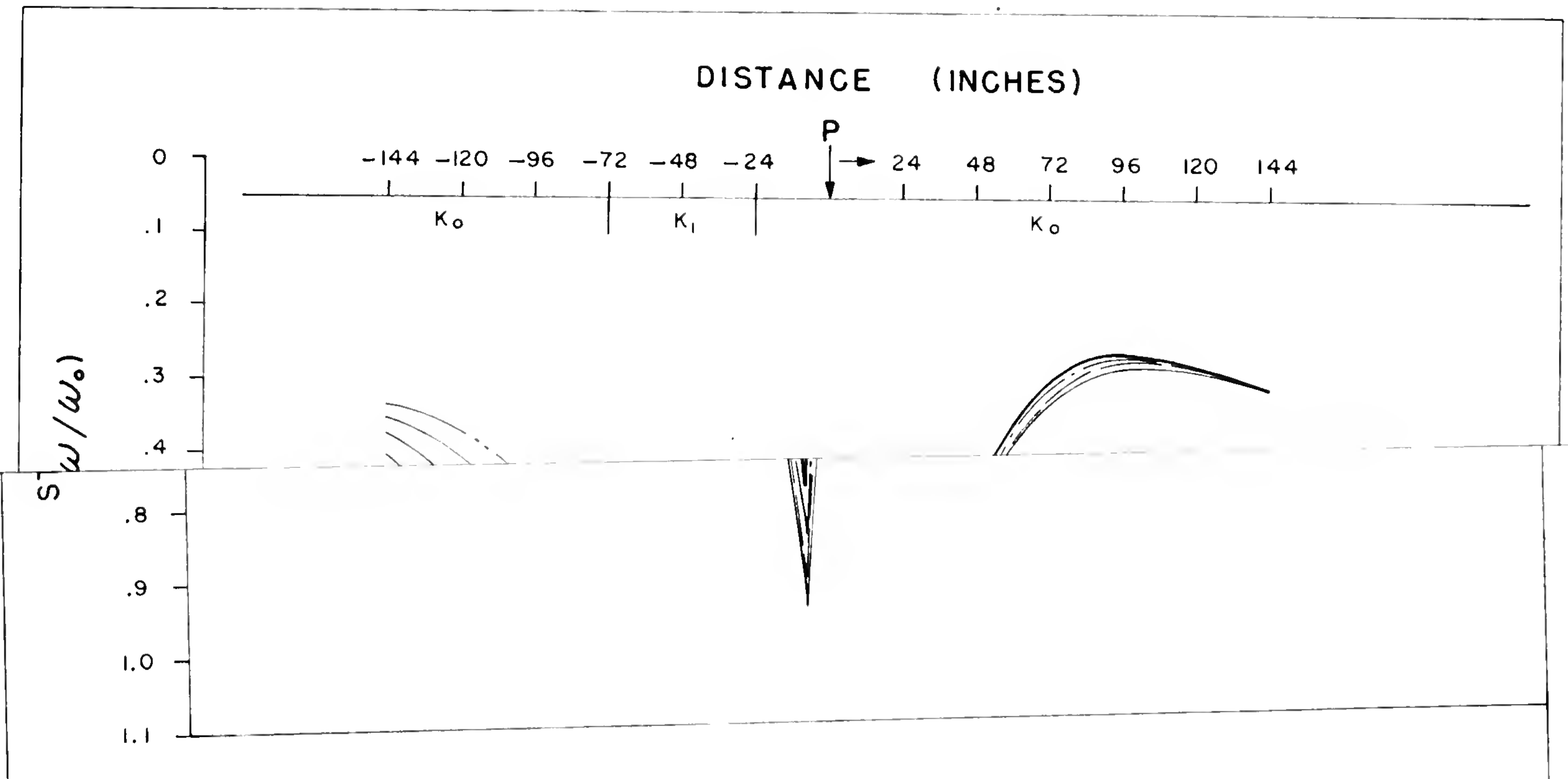


Figure 13. Deflection and Stress Amplification for Load 2 Ft. Ahead of Region of Reduced Subgrade Reaction

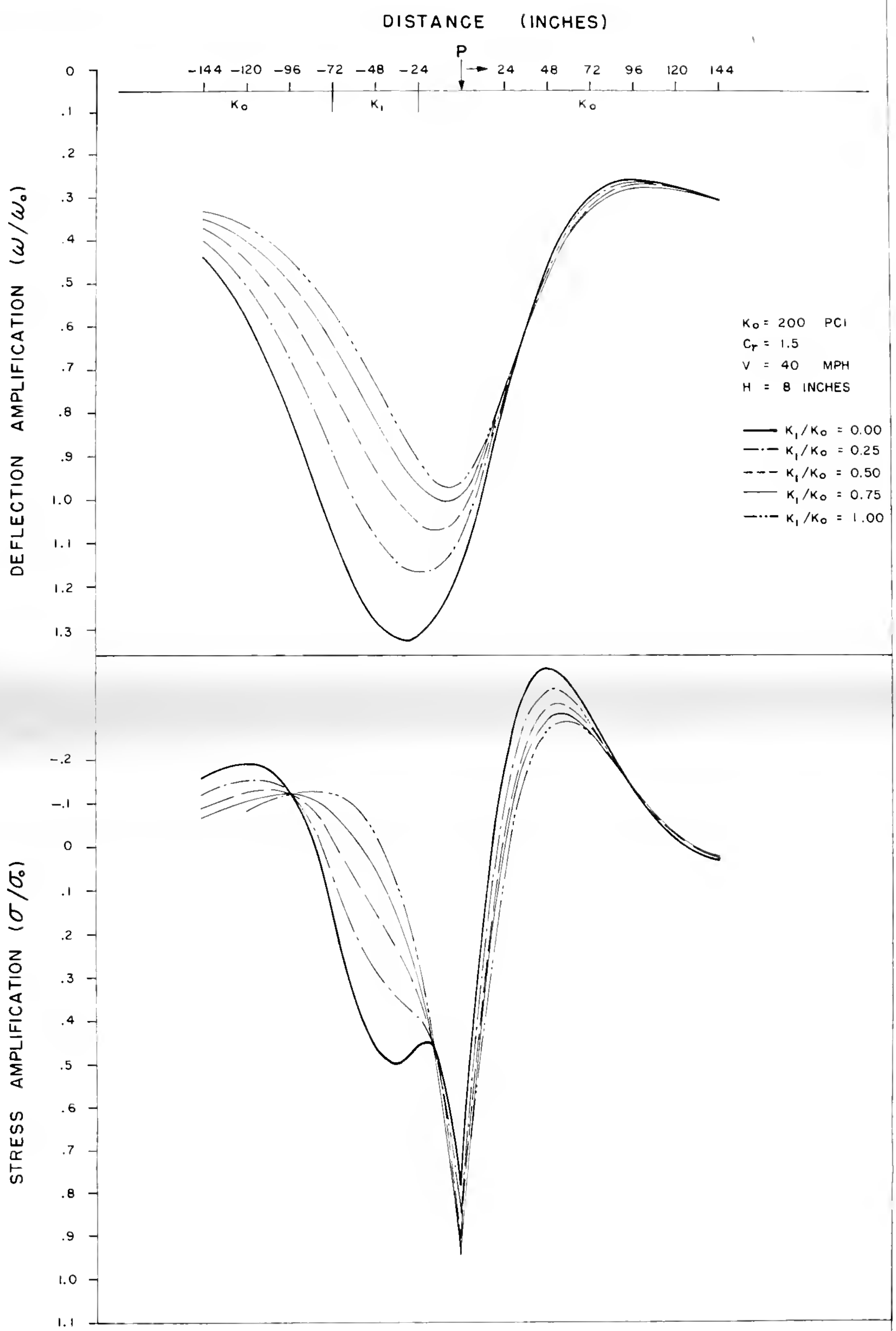


Figure 13. Deflection and Stress Amplification for Load 2 Ft. Ahead of Region of Reduced Subgrade Reaction

Table 1

Deflection and Stress Under a Static Load of 125 lbs/in

| | $K_o$ = 100 pci | | $K_o$ = 200 pci | |
|---|---|---|---|---|
| H ins | $w_o$ ins | $\sigma_o$ psi | $w_o$ ins | $\sigma_o$ ins |
| 8 | .019144 | 150.607 | .010722 | 126.645 |
| 10 | .019017 | 113.948 | .010481 | 95.819 |
| 12 | .019449 | 90.726 | .010573 | 76.291 |

and stress at a point. For example: For a pavement thickness of 8 inches and $K_o$ of 200 pci, the maximum stress under a static load of 125 lbs/in, is seen from Table 1 to be 126.645 psi. From Figure 12, for a velocity of 40 mph, $K_1/K_o = 0.5$ and $C_r = 1.5$, the stress amplification for a point 24 inches behind a load situated at the mid-point of the weakened region, is 0.431. The magnitude of the actual stress is then $0.431 \times 126.645 = 54.64$ psi.

Figures 14 and 15 illustrate the variation of maximum stress and deflection with velocity, while Figures 16, 17, 18 and 19 reveal the influence of the parametric length, $\ell$. Similar plots are presented in Figures 20 through 23 for the parameter, H.

Curves similar to those given in Figures 11, 12 and 13 are presented in Figures 24, 25 and 26 for the special case where there is only 50% load transfer and no moment transfer across a discontinuity in the pavement's surface.

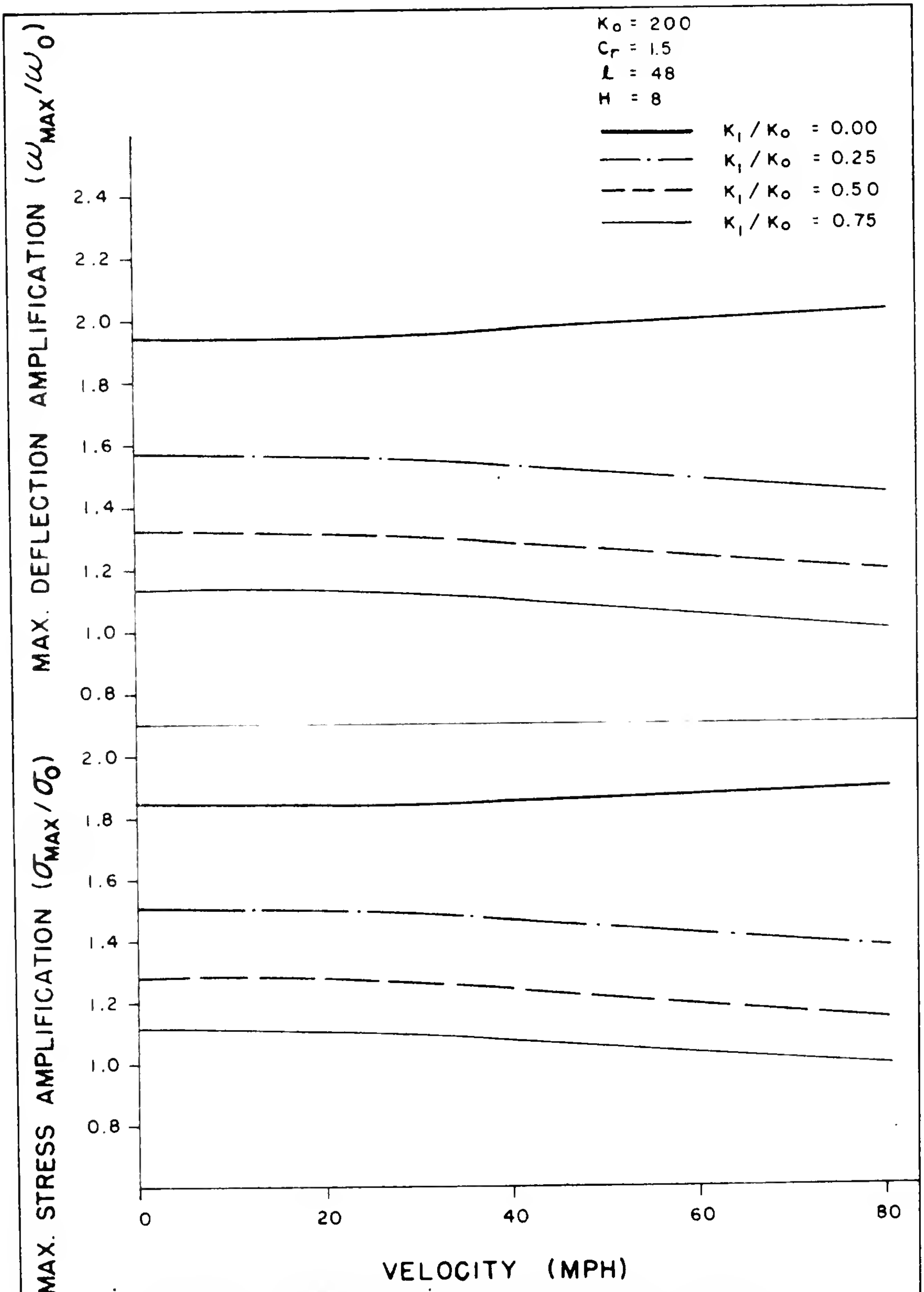


Figure 14a Maximum Deflection and Stress Amplification vrs Velocity ($K_o$= 200 pci., $C_r$=1.5)

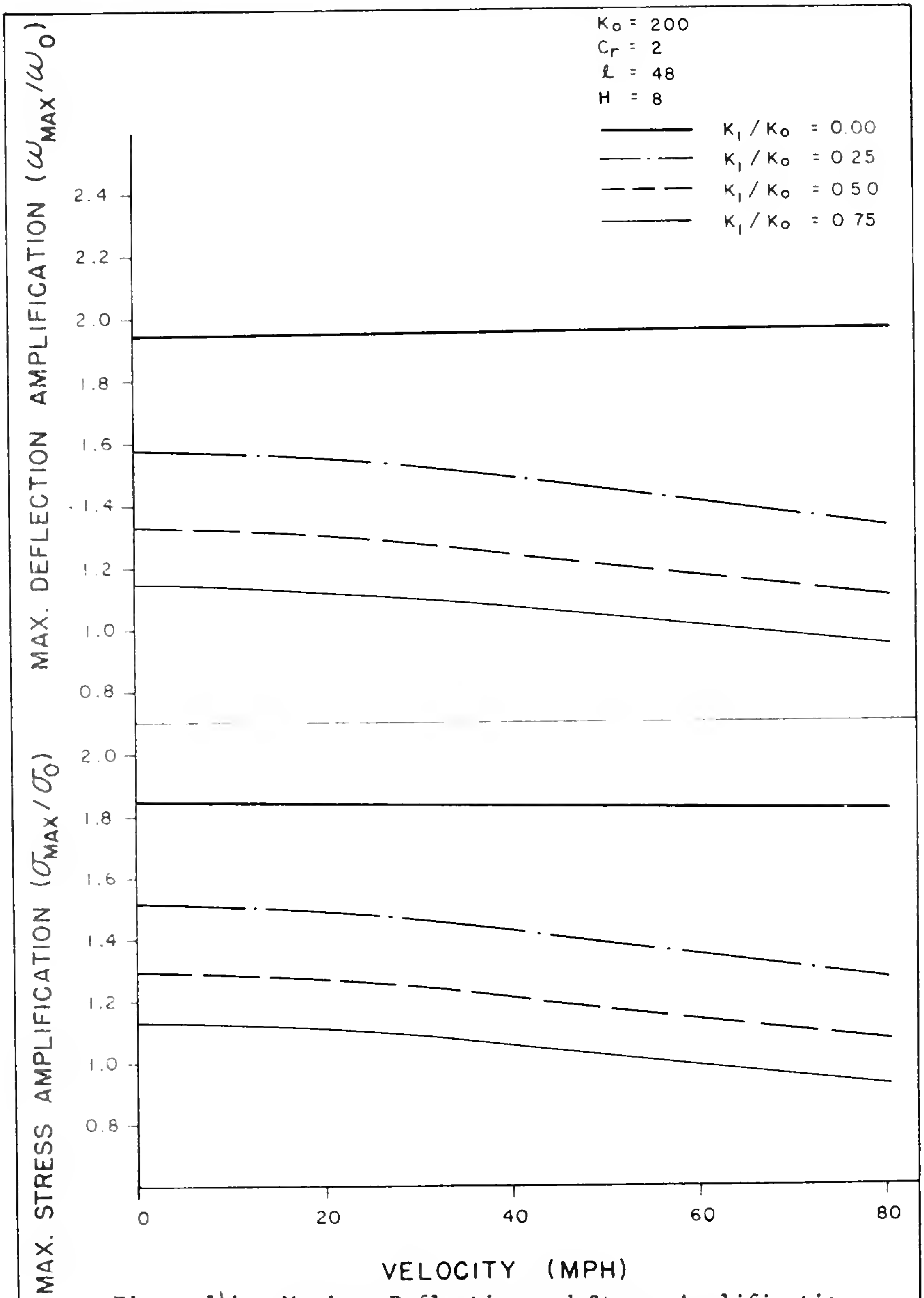


Figure 14b. Maximum Deflection and Stress Amplification vrs Velocity ($K_o$=200 pci., $C_r$=2.0)

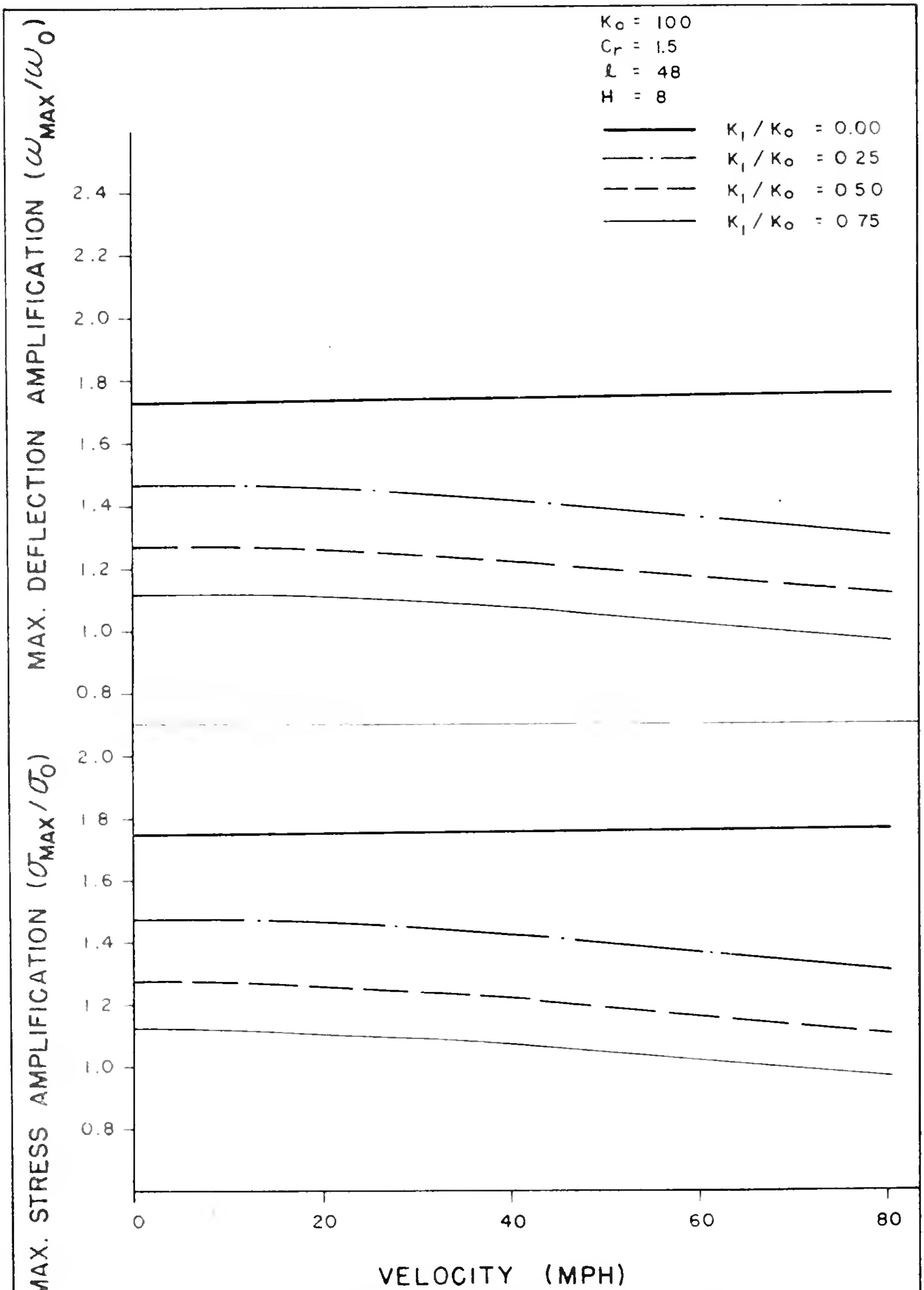


Figure 15a. Maximum Deflection and Stress Amplification vrs Velocity ($K_o$=100 pci., $C_r$=1.5)

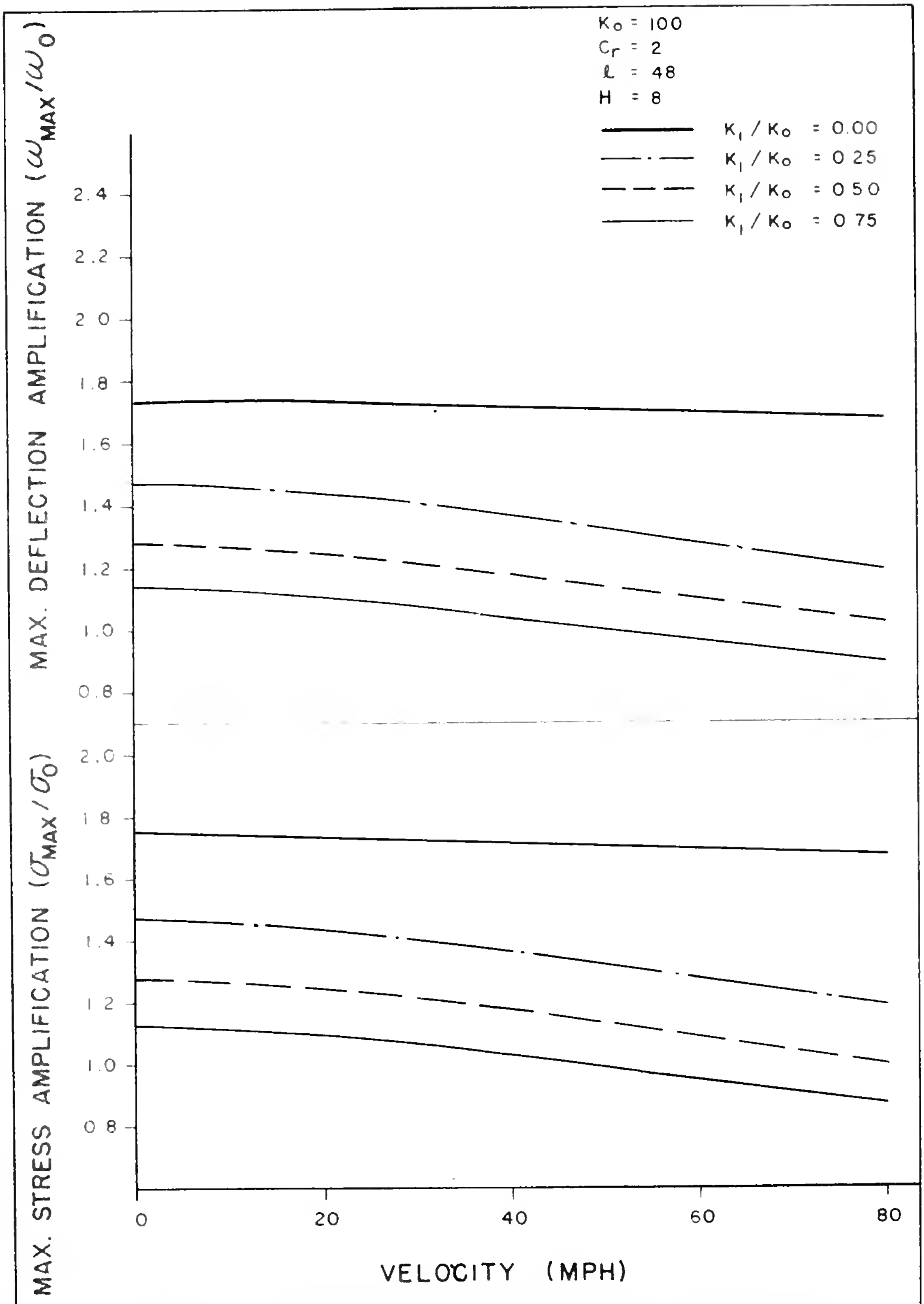


Figure 15b. Maximum Deflection and Stress Amplification vrs Velocity ($K_o$=100 pci., $C_r$=2.0)

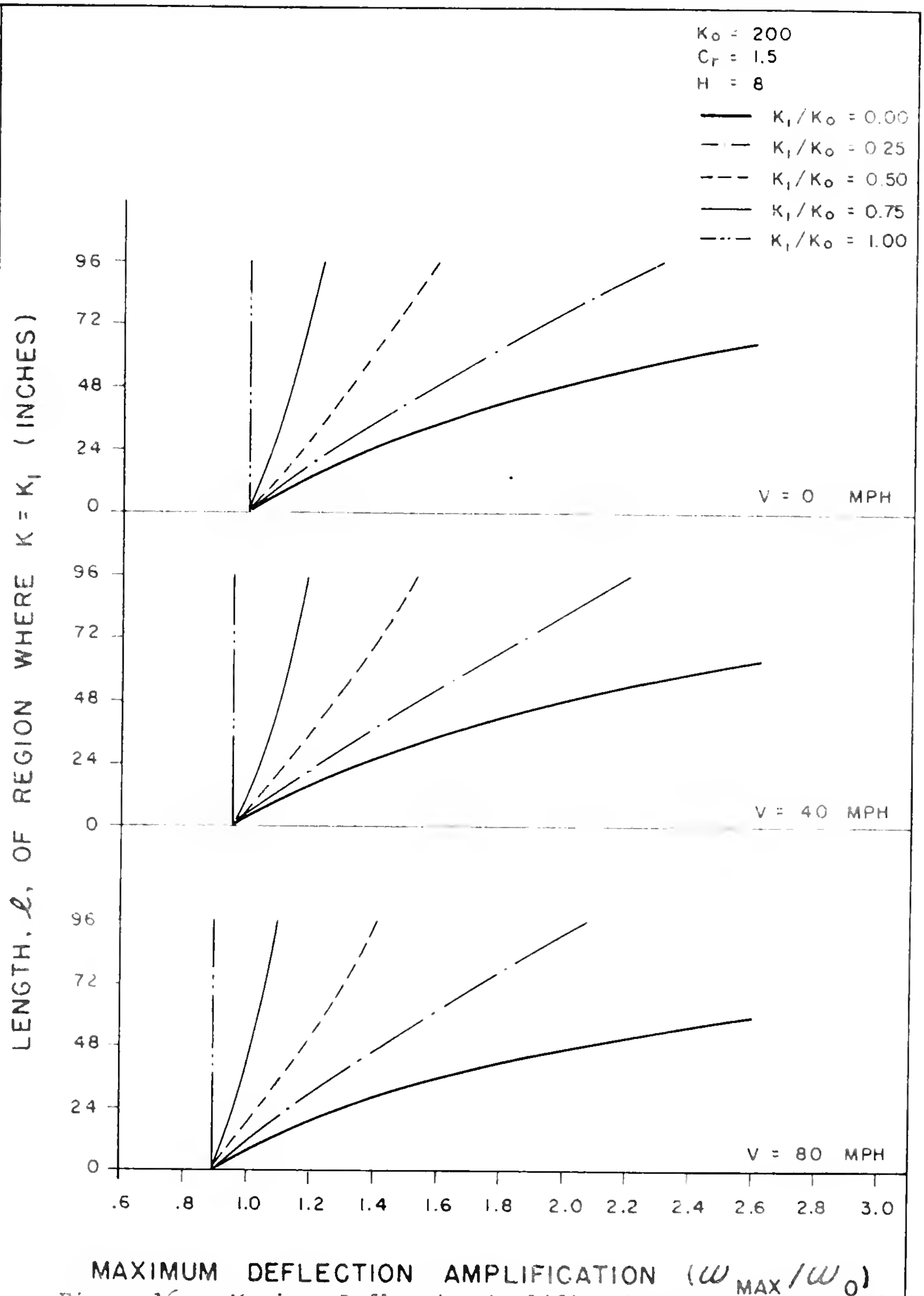


Figure 16a. Maximum Deflection Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=200 pci., $C_r$=1.5)

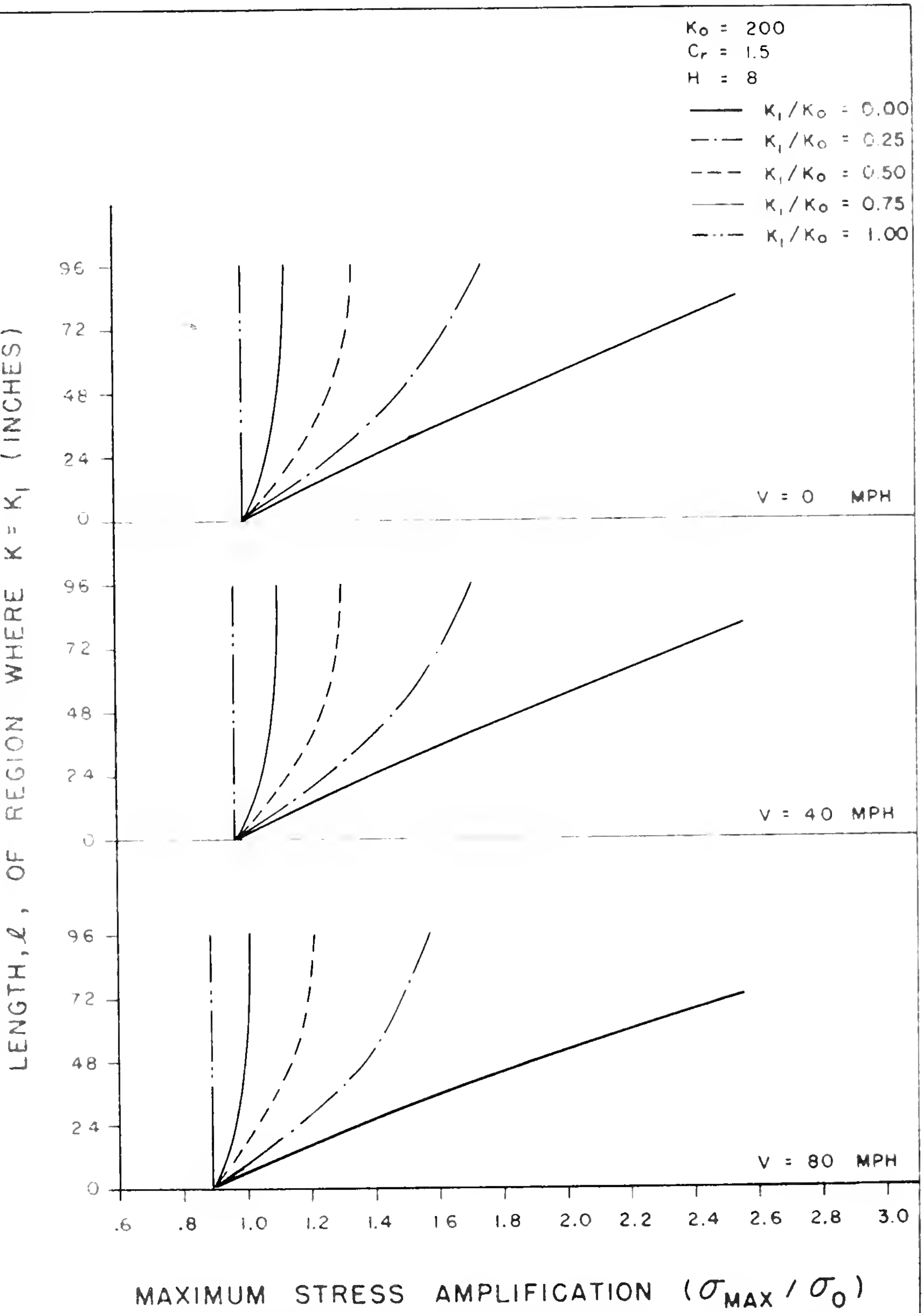


Figure 16b. Maximum Stress Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=200 pci., $C_r$=1.5)

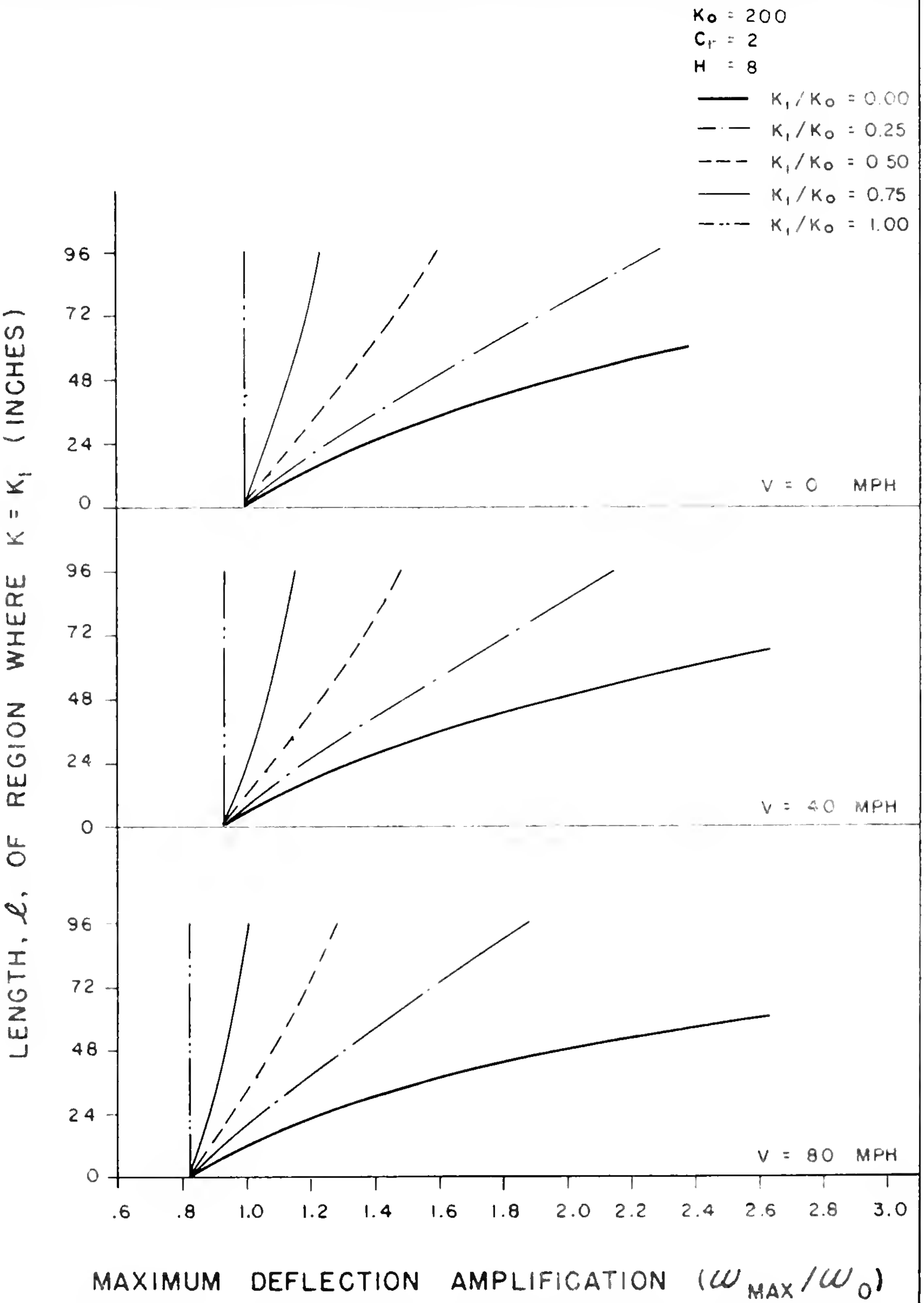


Figure 17a. Maximum Deflection Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=200 pci., $C_r$=2.0)

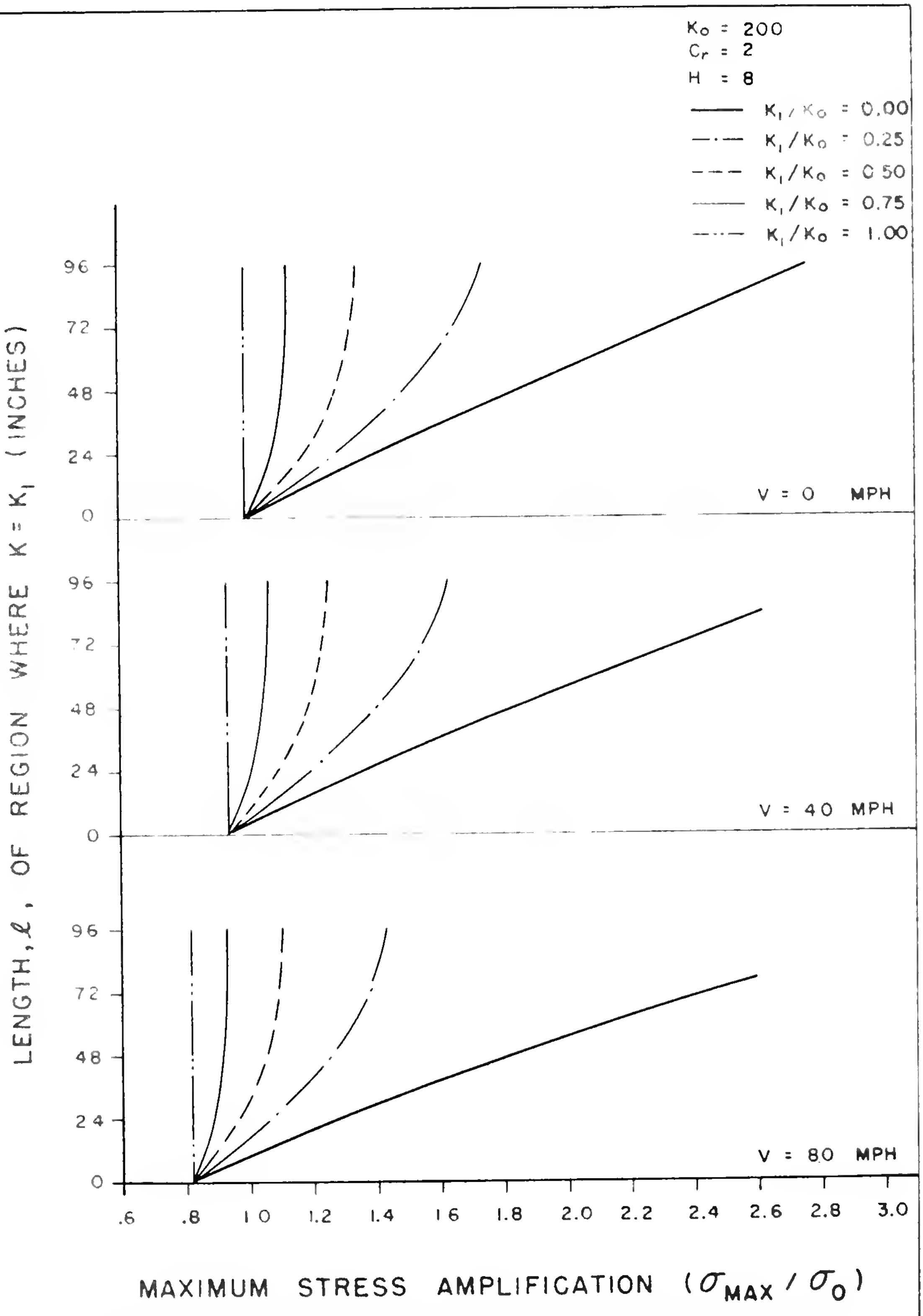


Figure 17b. Maximum Stress Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=200 pci., $C_r$=2.0)

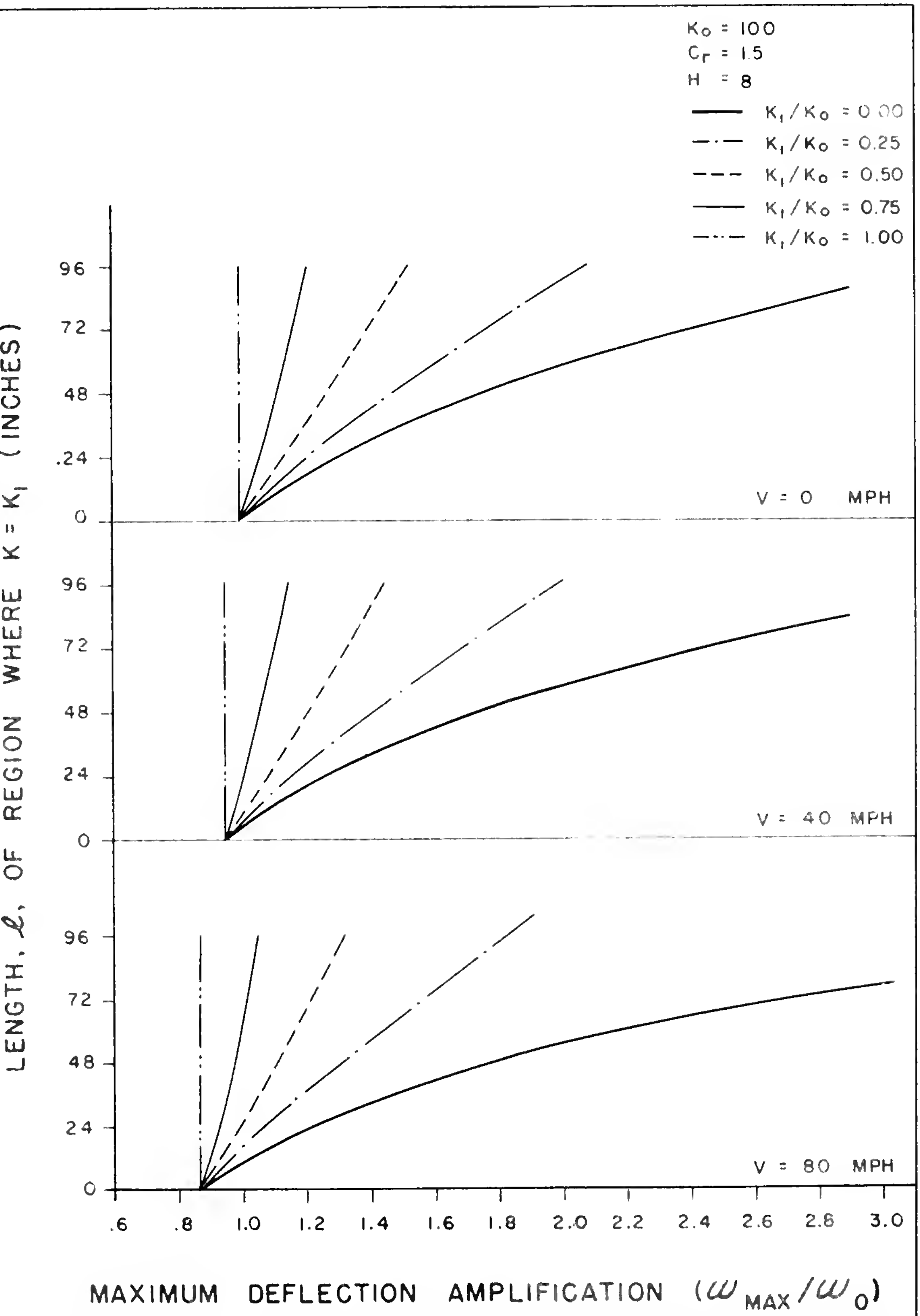


Figure 18a. Maximum Deflection Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=100 pci., $C_r$=1.5)

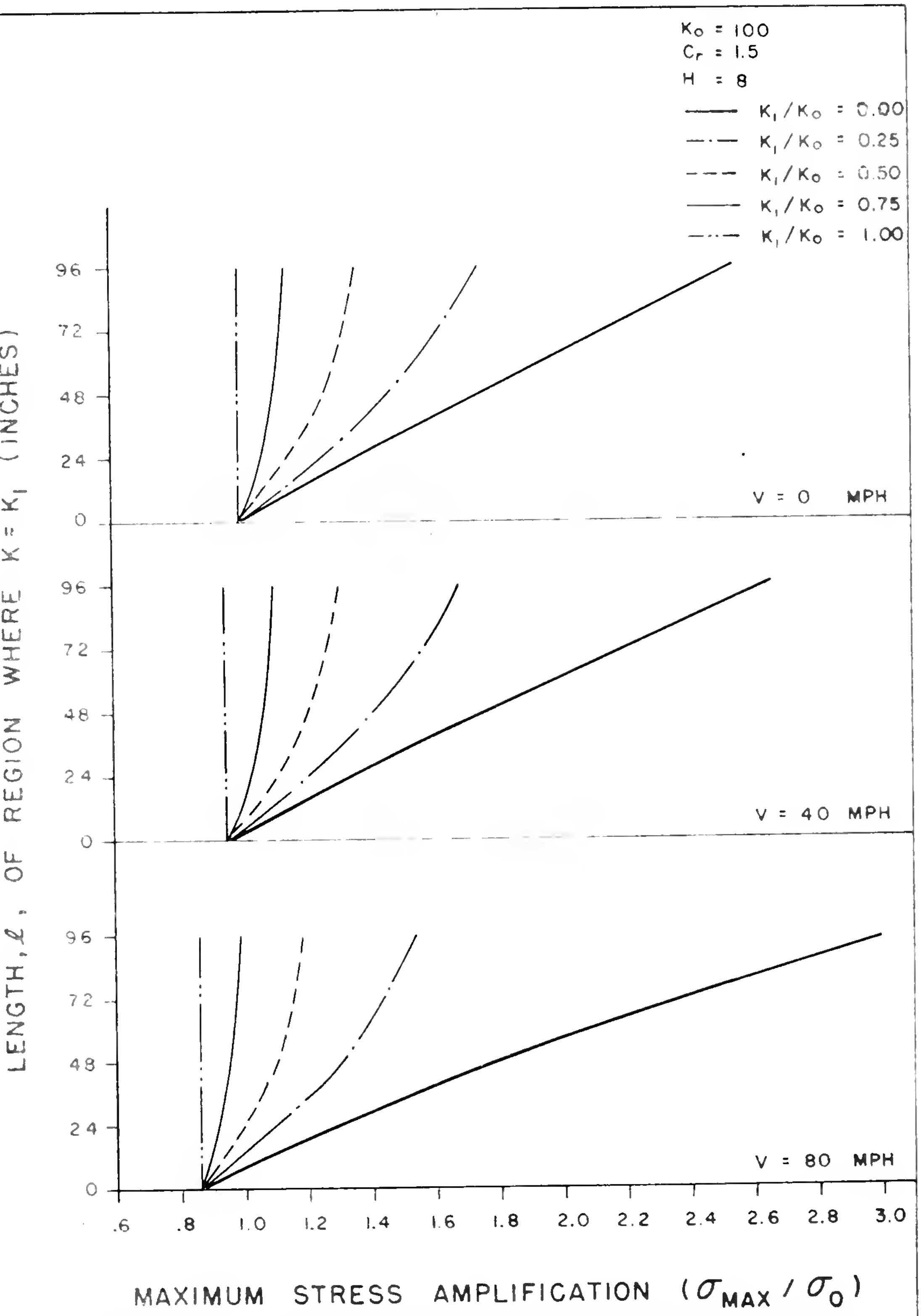


Figure 18b. Maximum Stress Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=100 pci., $C_r$=1.5)

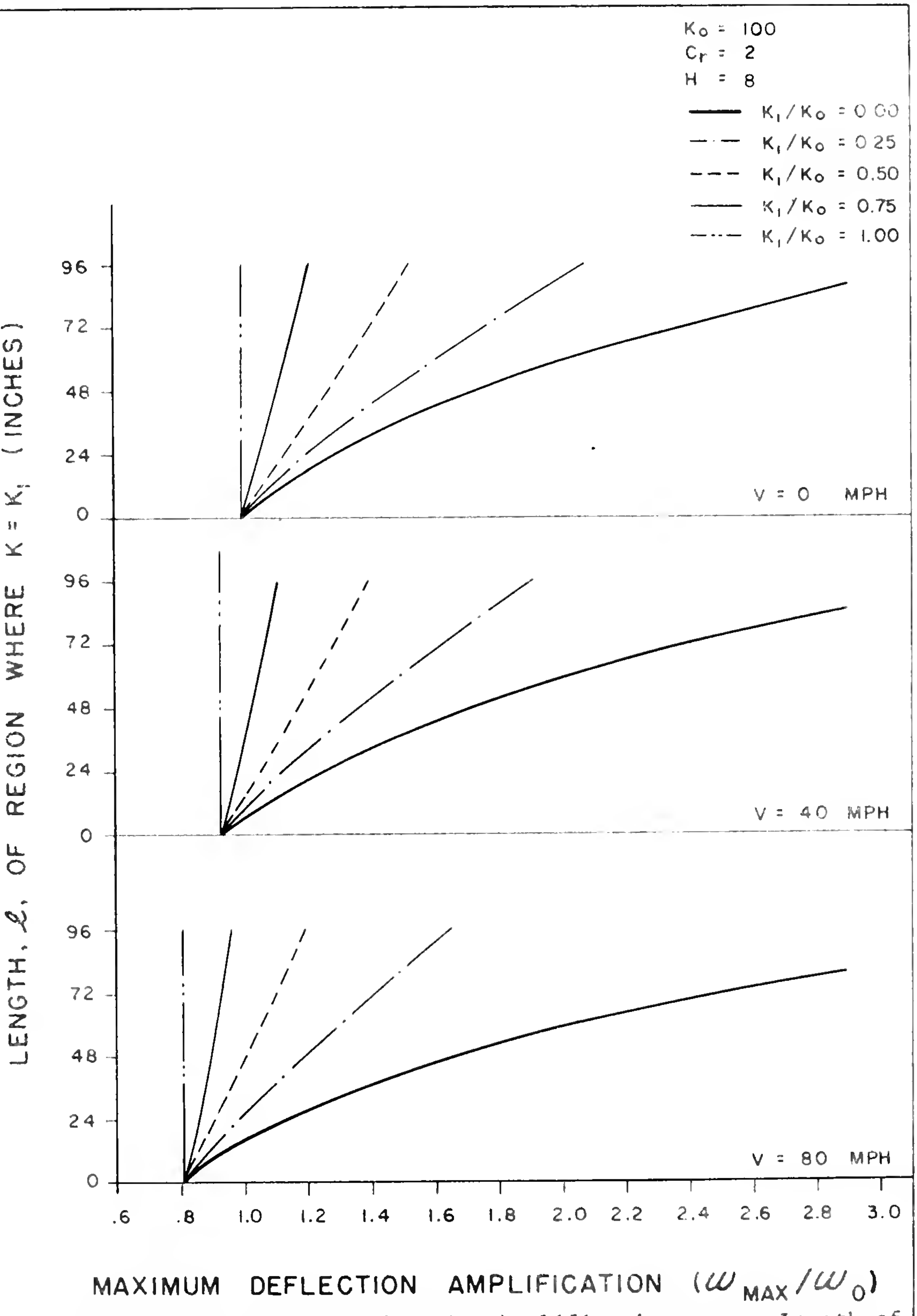


Figure 19a. Maximum Deflection Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=100 pci., $C_r$=2.0)

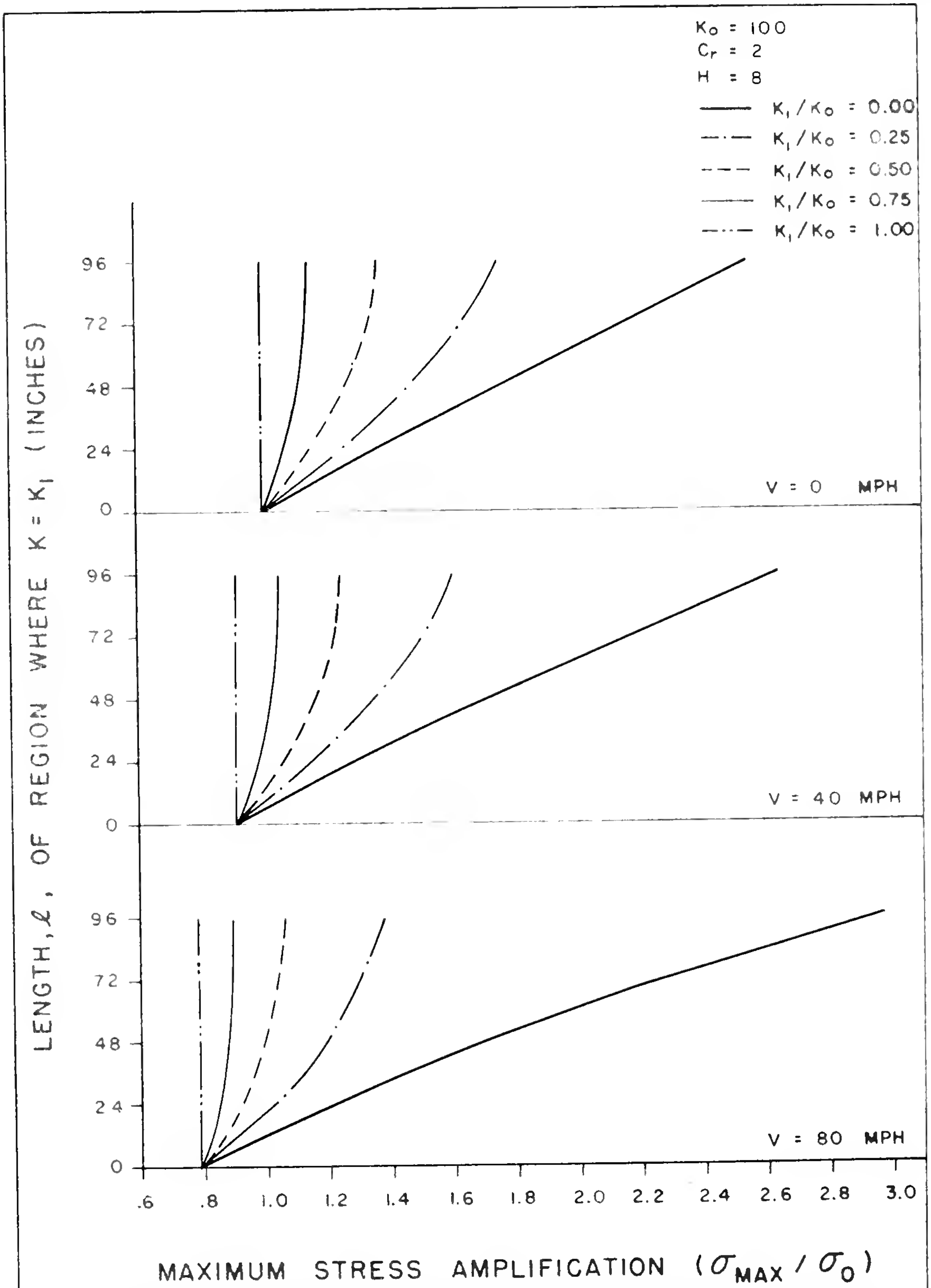


Figure 19b. Maximum Stress Amplification versus Length of Region of Reduced Subgrade Reaction ($K_o$=100 pci., $C_r$=2.0)

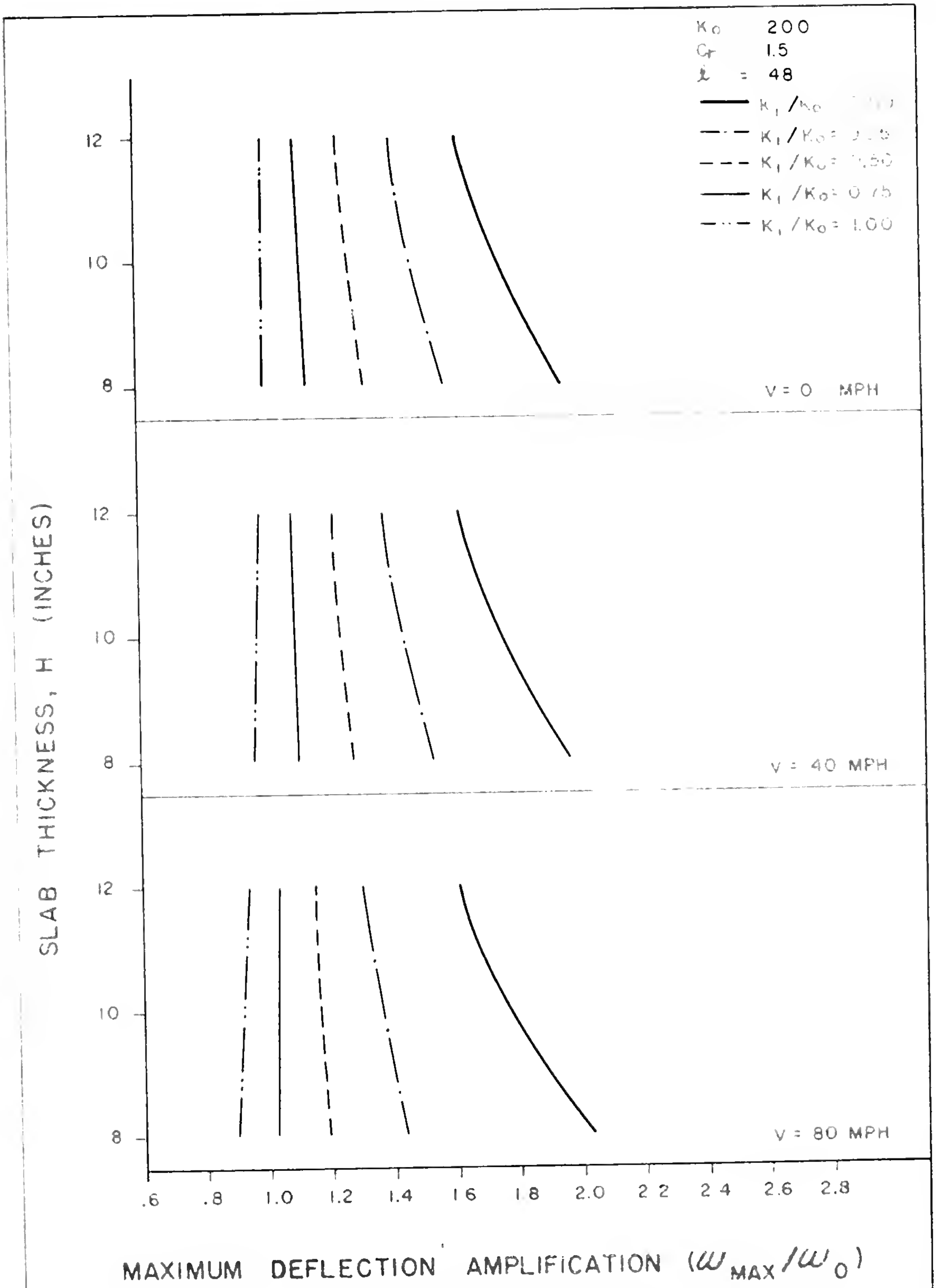


Figure 20a. Maximum Deflection Amplification versus Slab Thickness ($K_o$=200 pci., $C_r$=1.5)

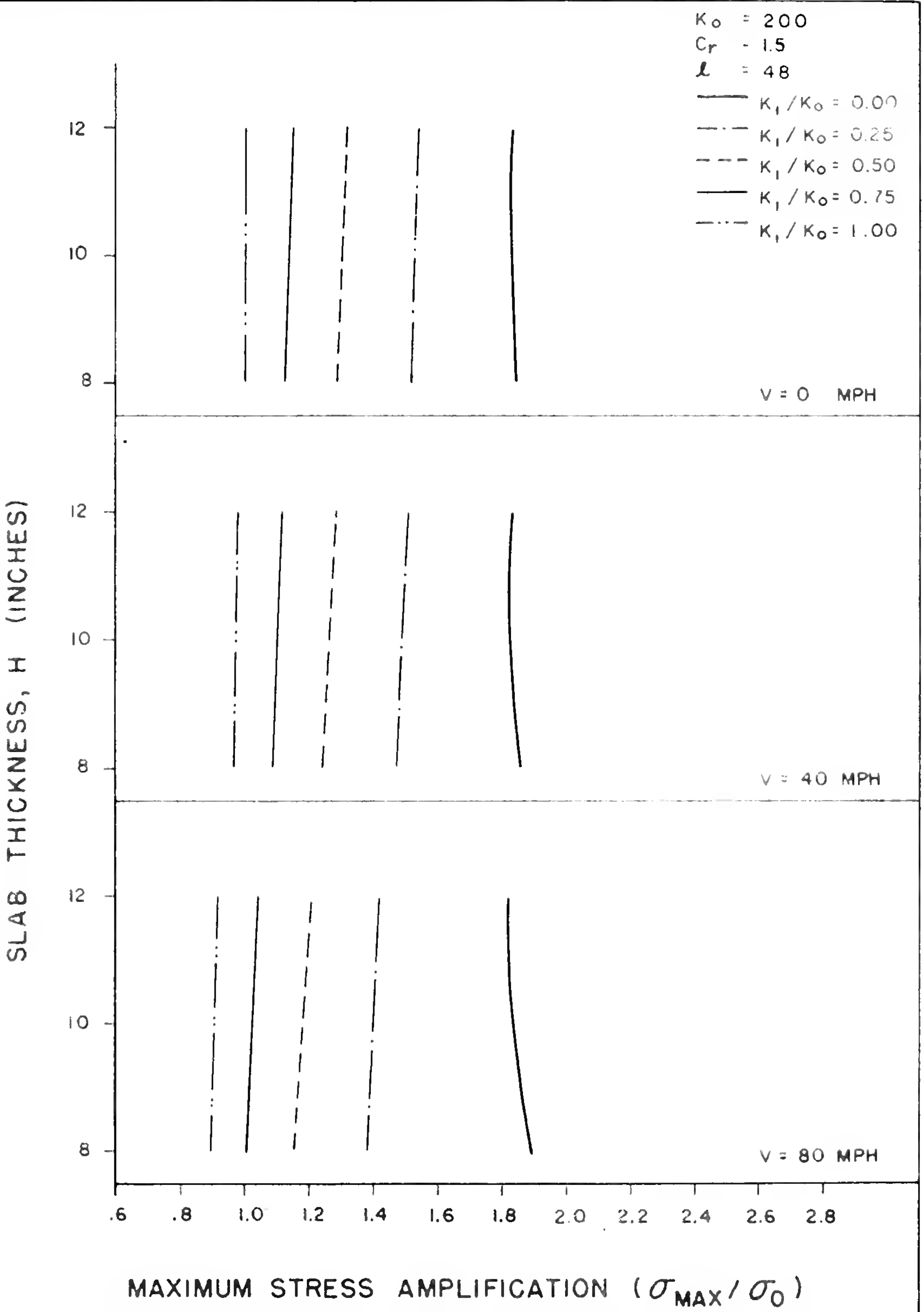


Figure 20b. Maximum Stress Amplification versus Slab Thickness ($K_o$=200 pci., $C_r$=1.5)

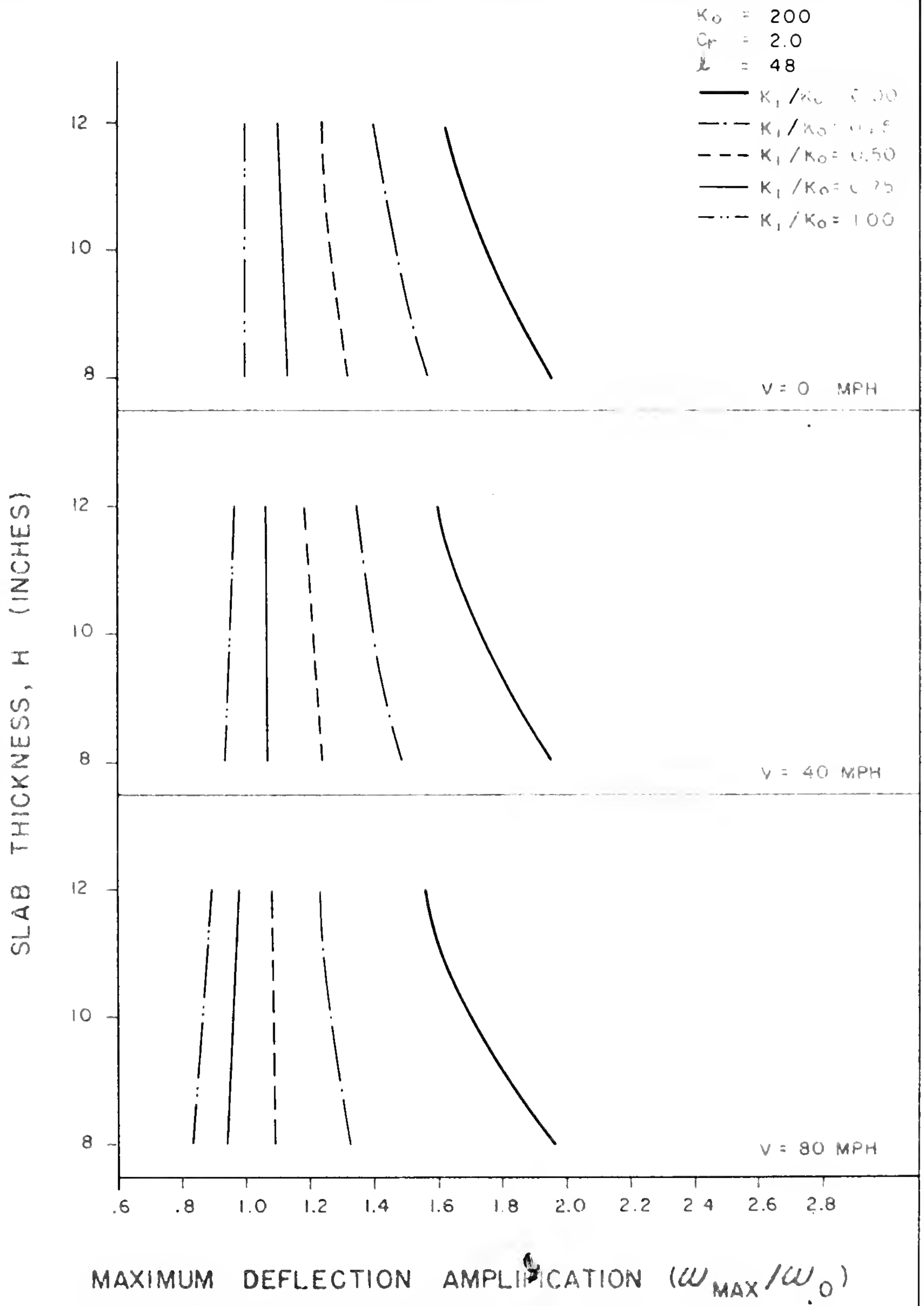


Figure 21a. Maximum Deflection Amplification versus Slab Thickness ($K_o$=200 pci., $C_r$=2.0)

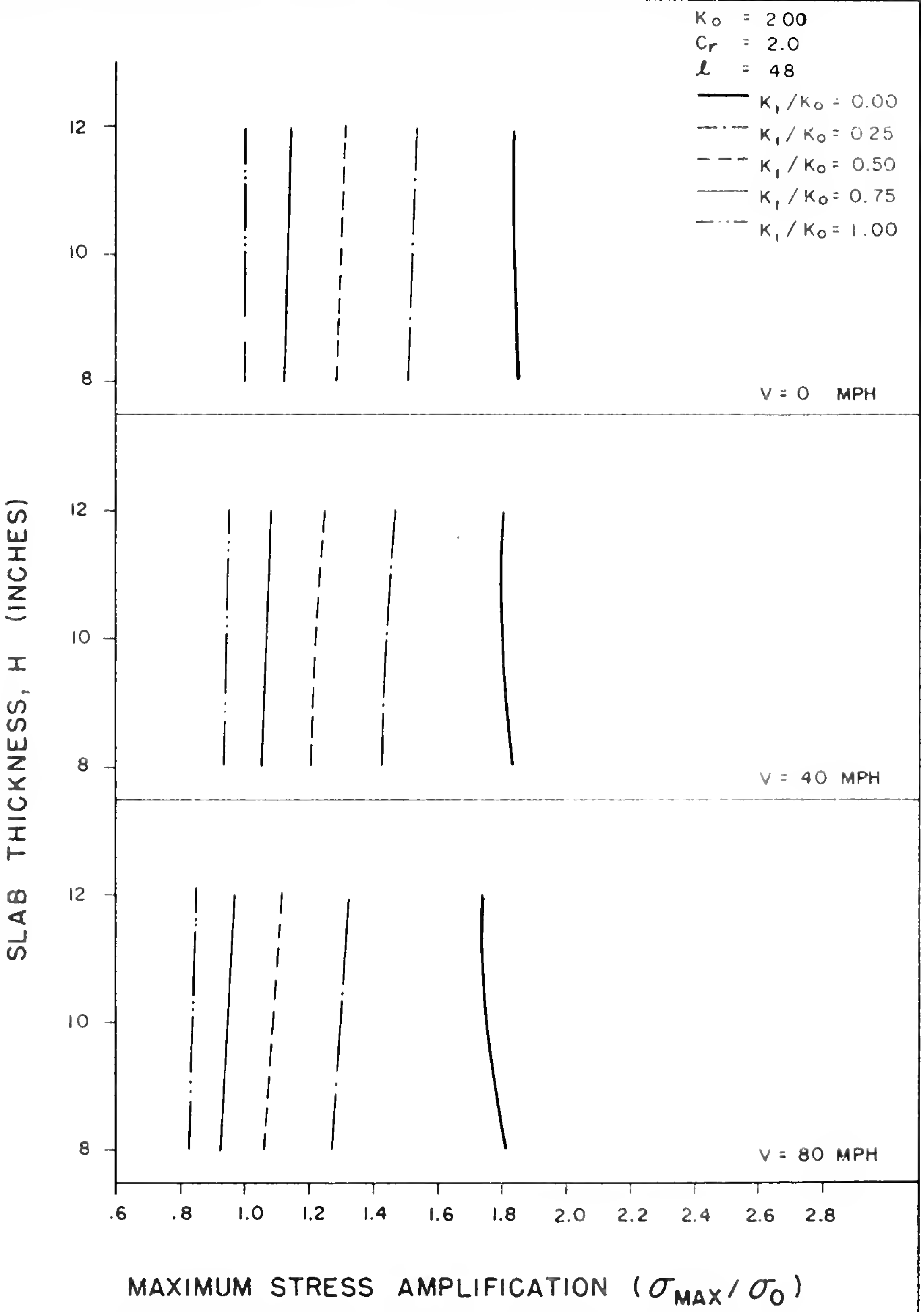


Figure 21b. Maximum Stress Amplification versus Slab Thickness ($K_o$=200 pci., $C_r$=2.0)

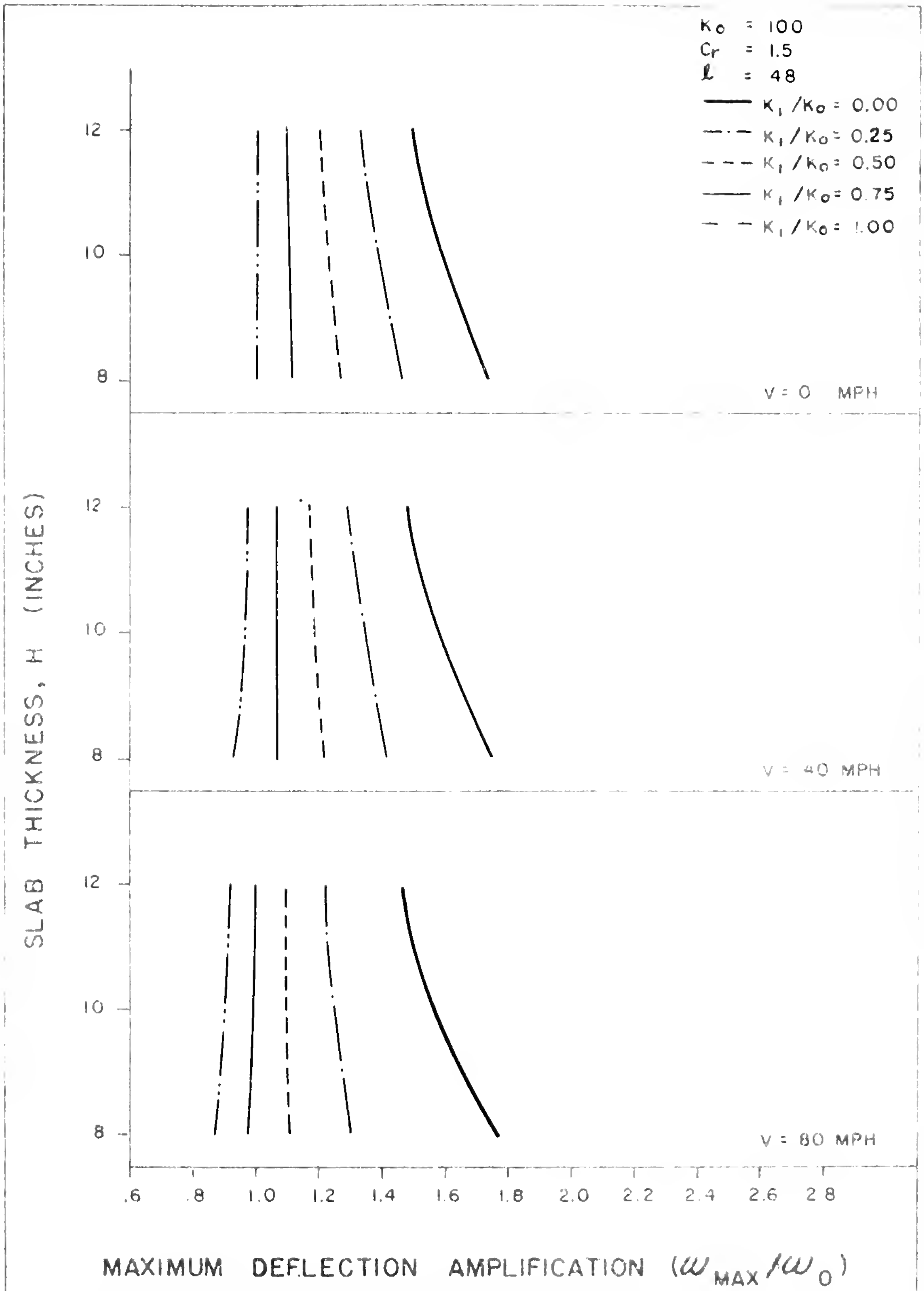


Figure 22a. Maximum Deflection Amplification versus Slab Thickness ($K_o$=100 pci., $C_r$=1.5)

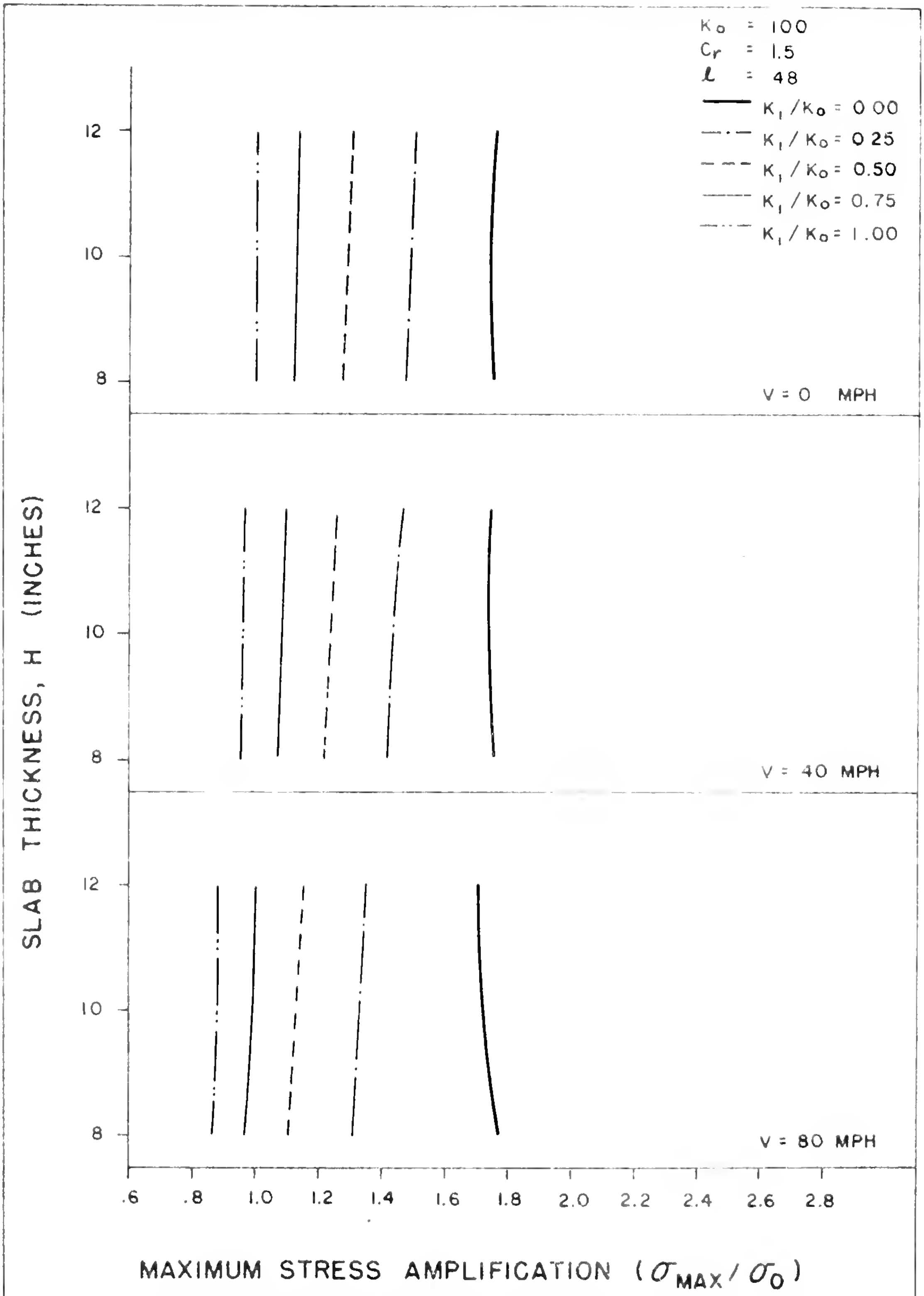


Figure 22b. Maximum Stress Amplification versus Slab Thickness ($K_o$=100 pci., $C_r$=1.5)

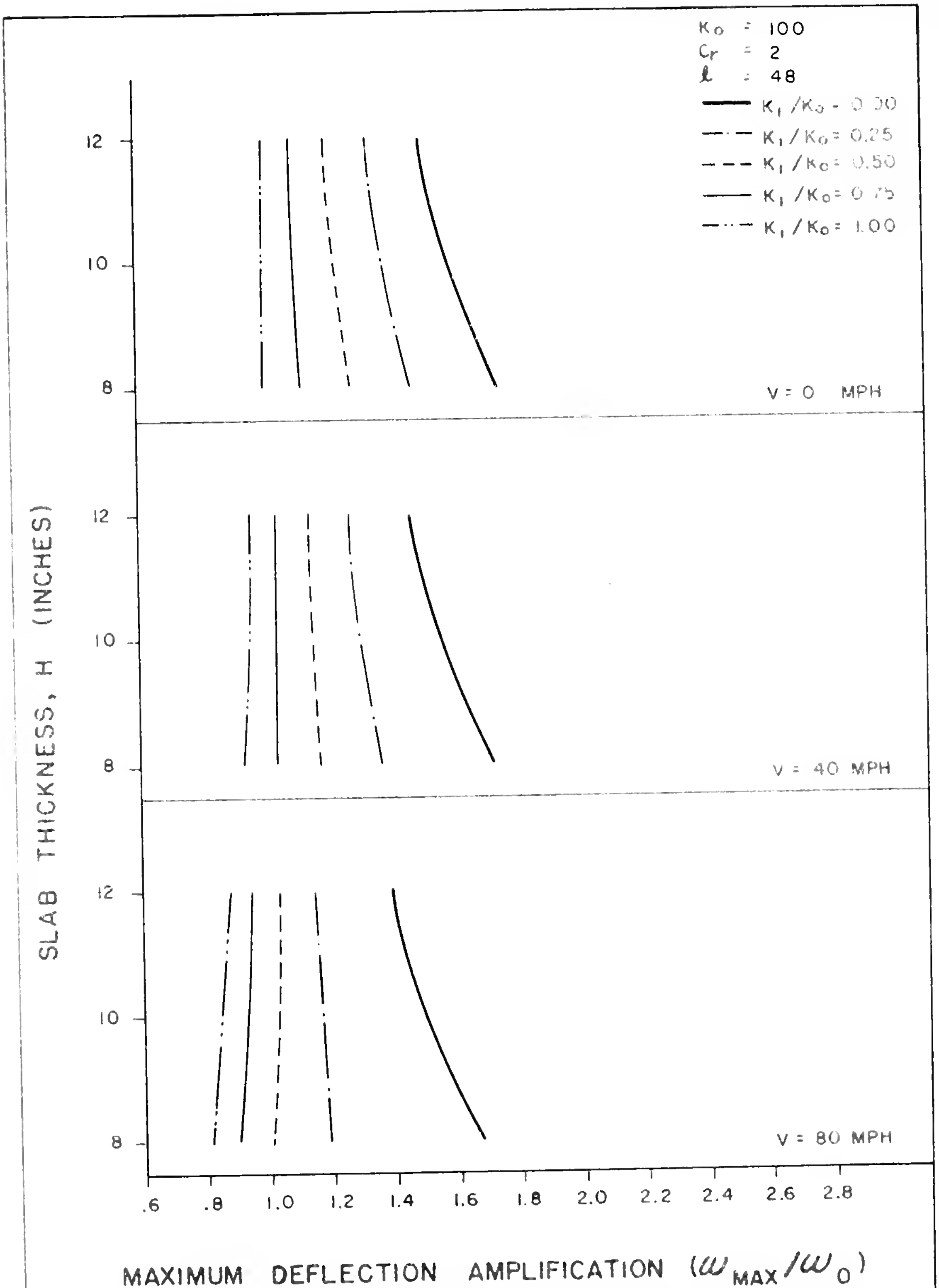


Figure 23a. Maximum Deflection Amplification versus Slab Thickness ($K_o$=100 pci., $C_r$=2.0)

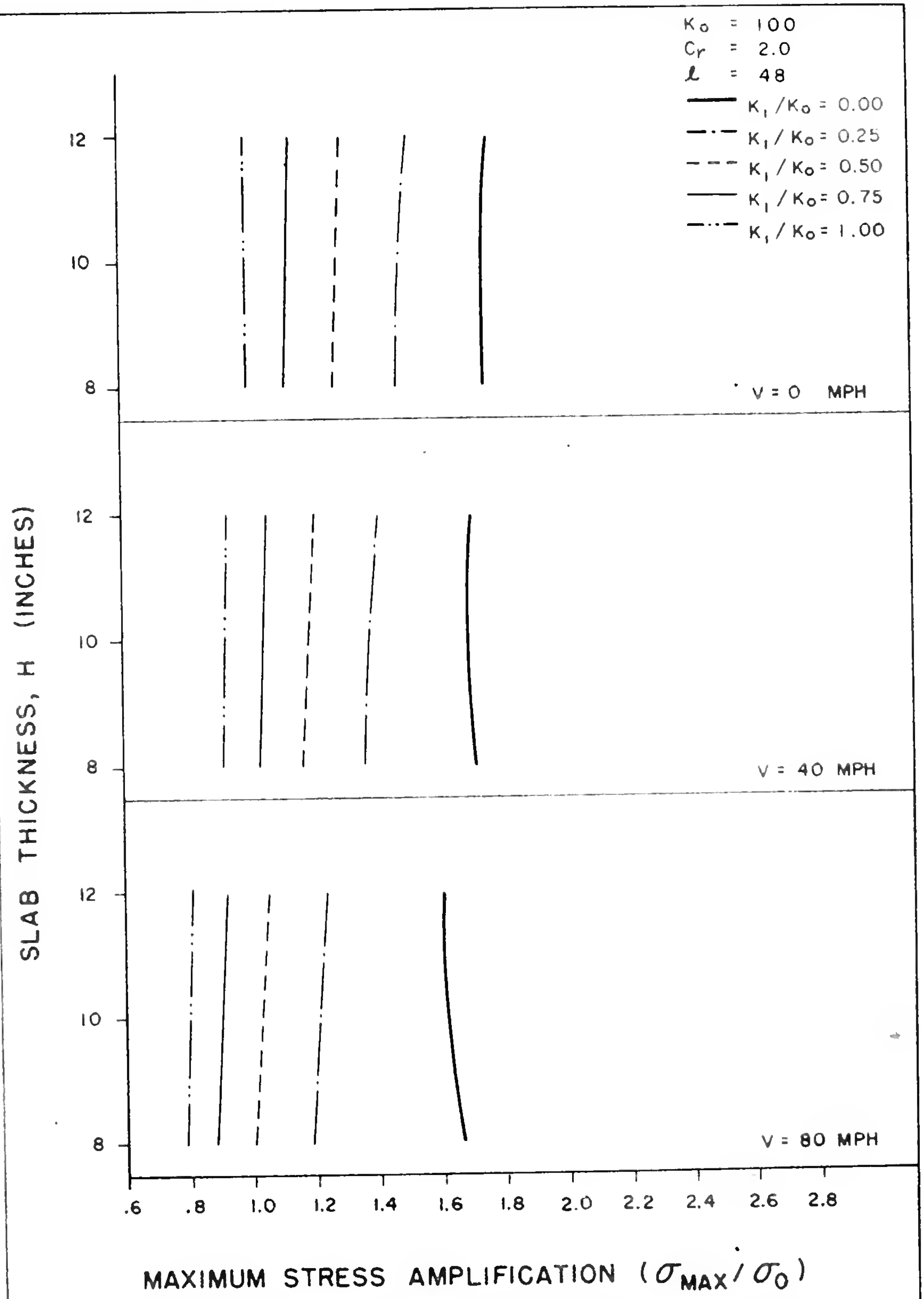


Figure 23b. Maximum Stress Amplification versus Slab Thickness ($K_o$=100 pci., $C_r$=2.0)

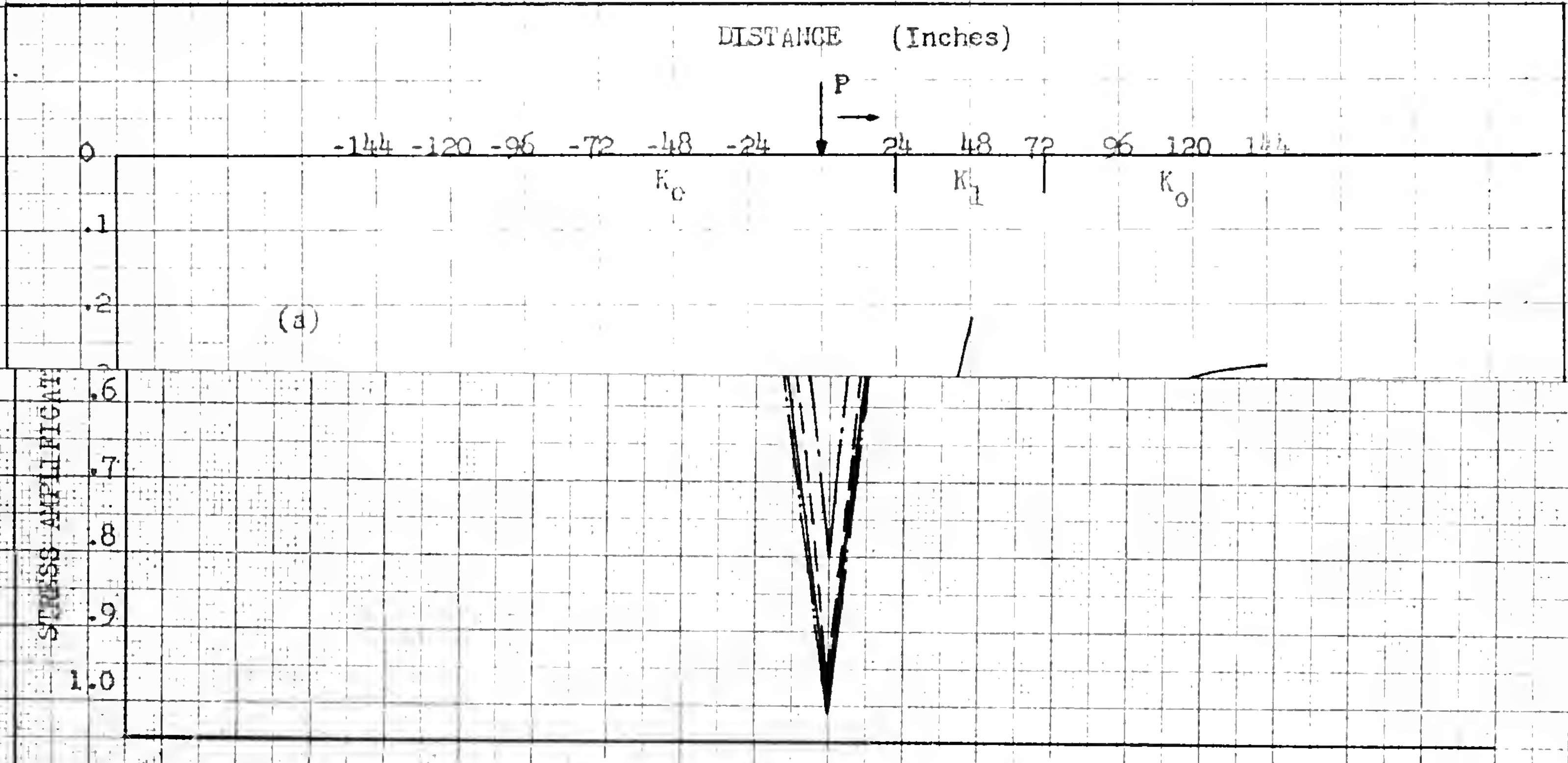


Figure 24. Deflection and Stress Amplification for Load 2 feet Behind Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

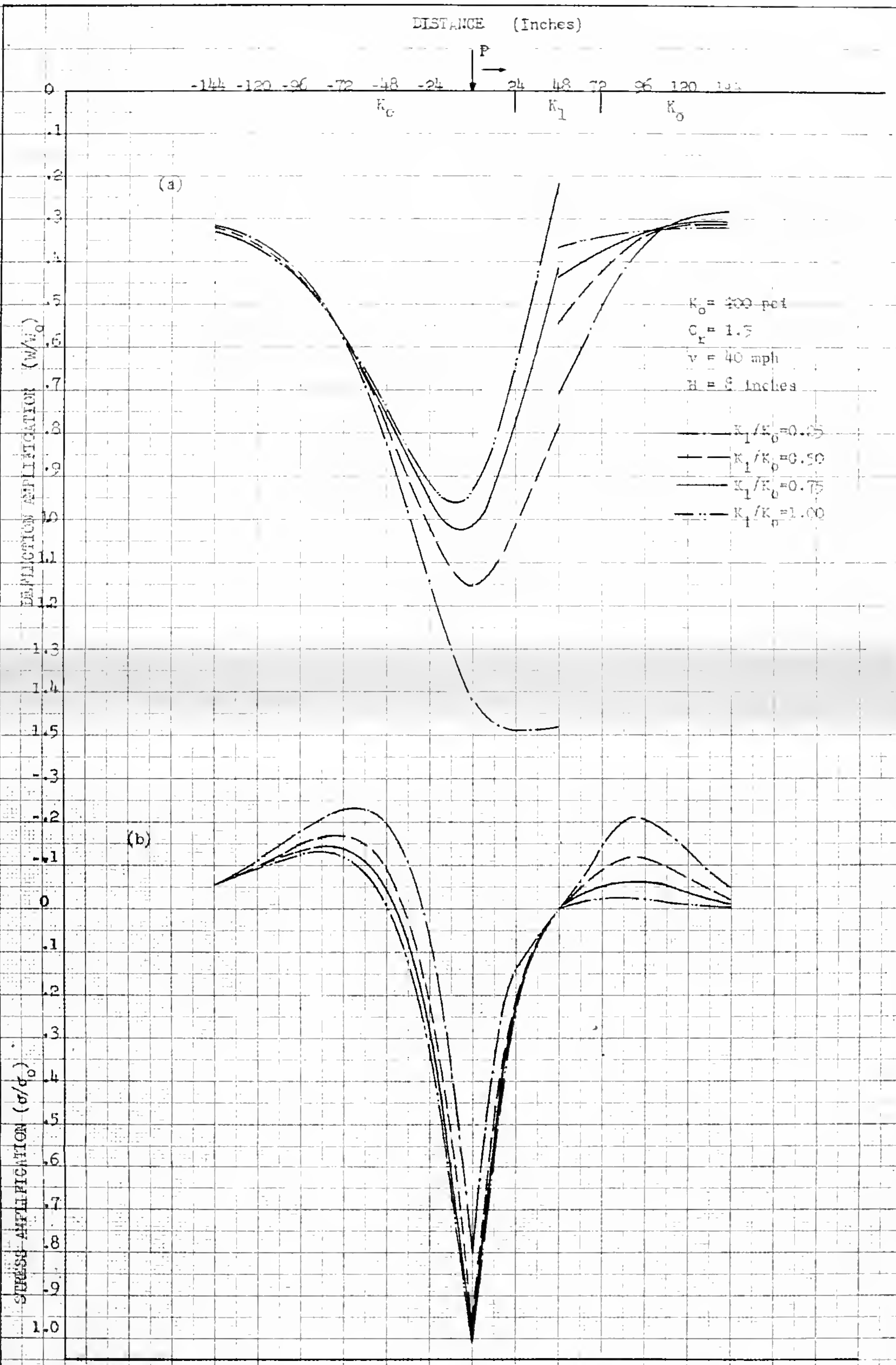


Figure 24. Deflection and Stress Amplification for Load 2 feet Behind Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

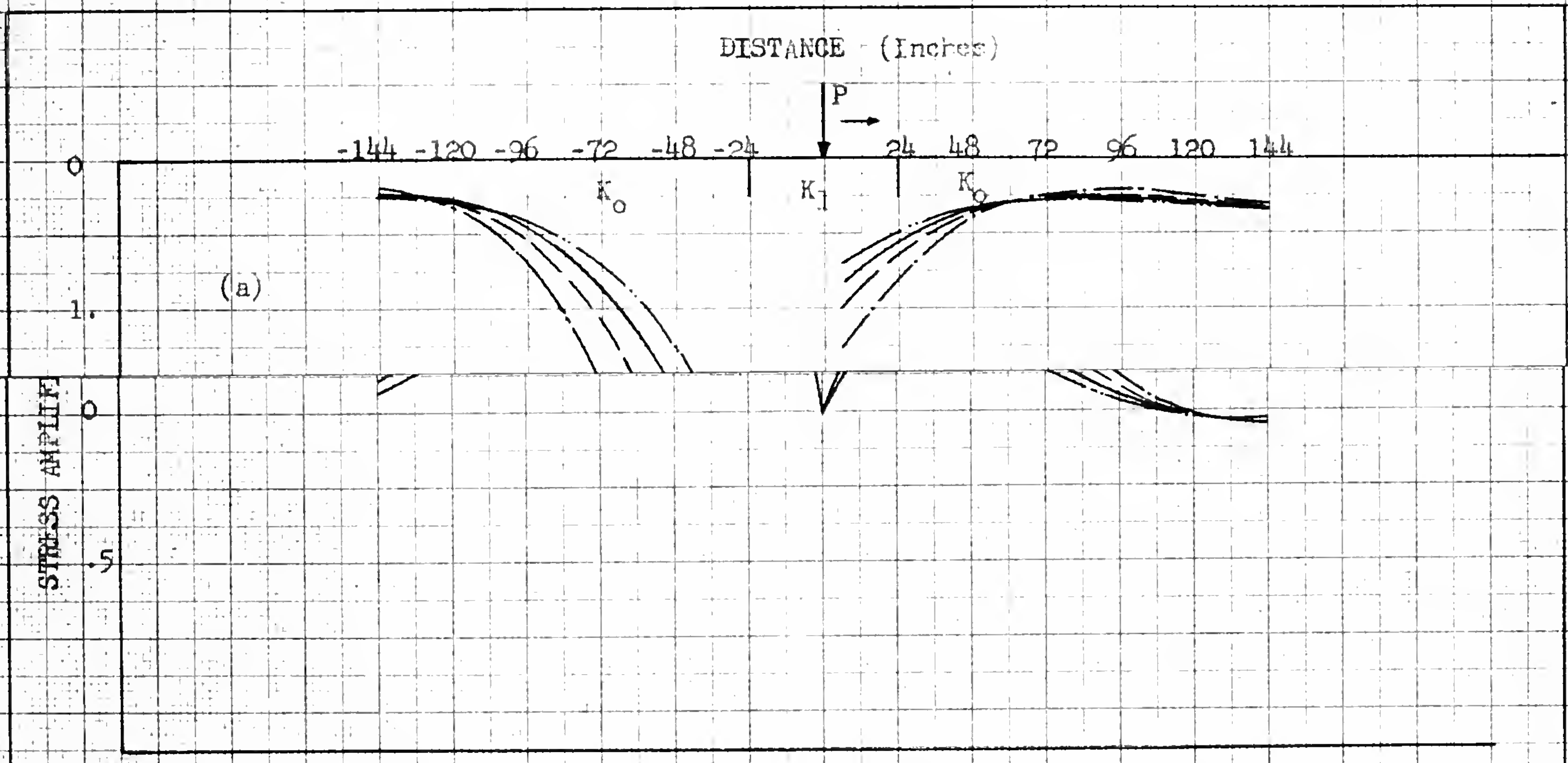


Figure 25. Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

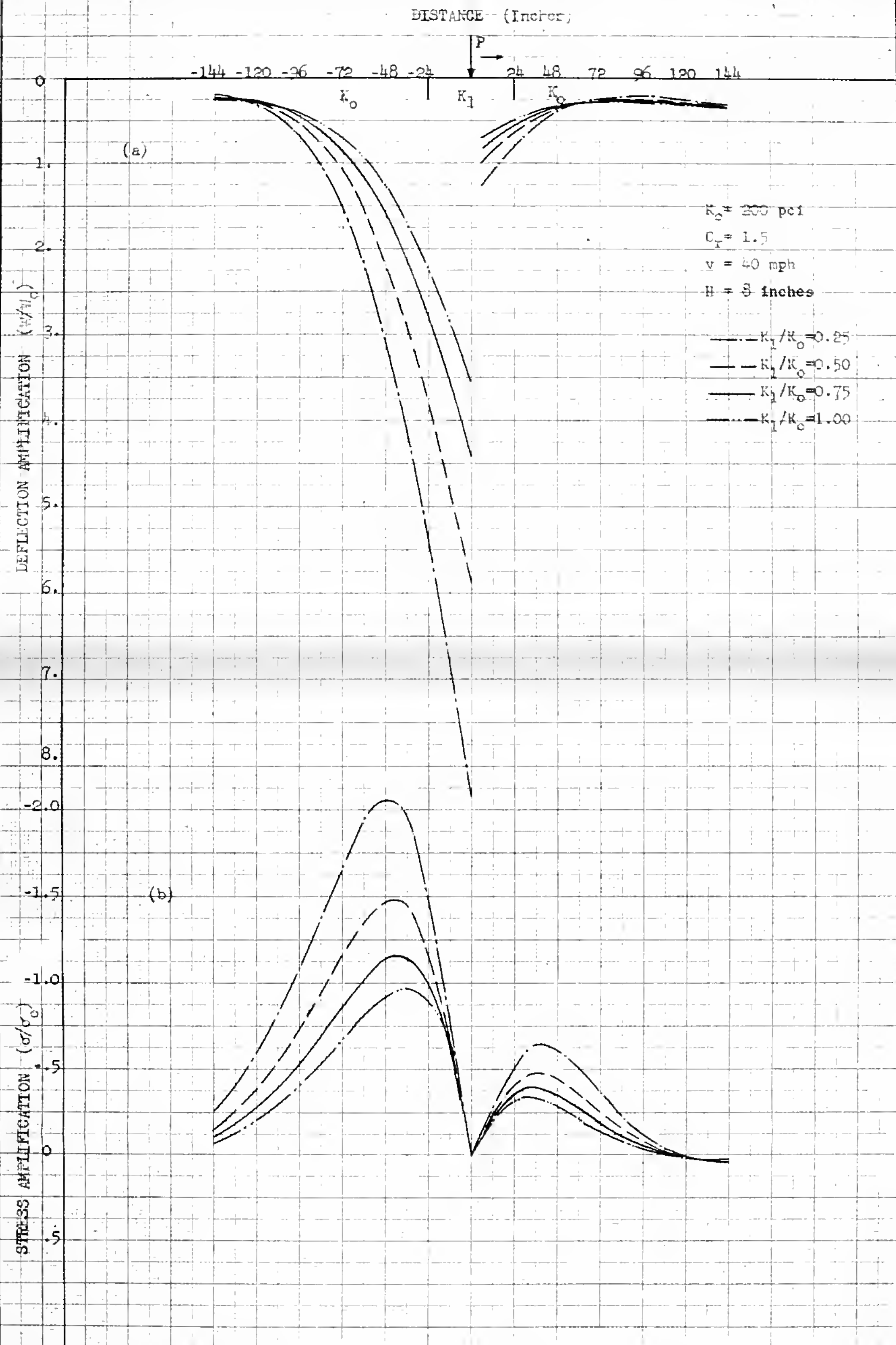


Figure 25. Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

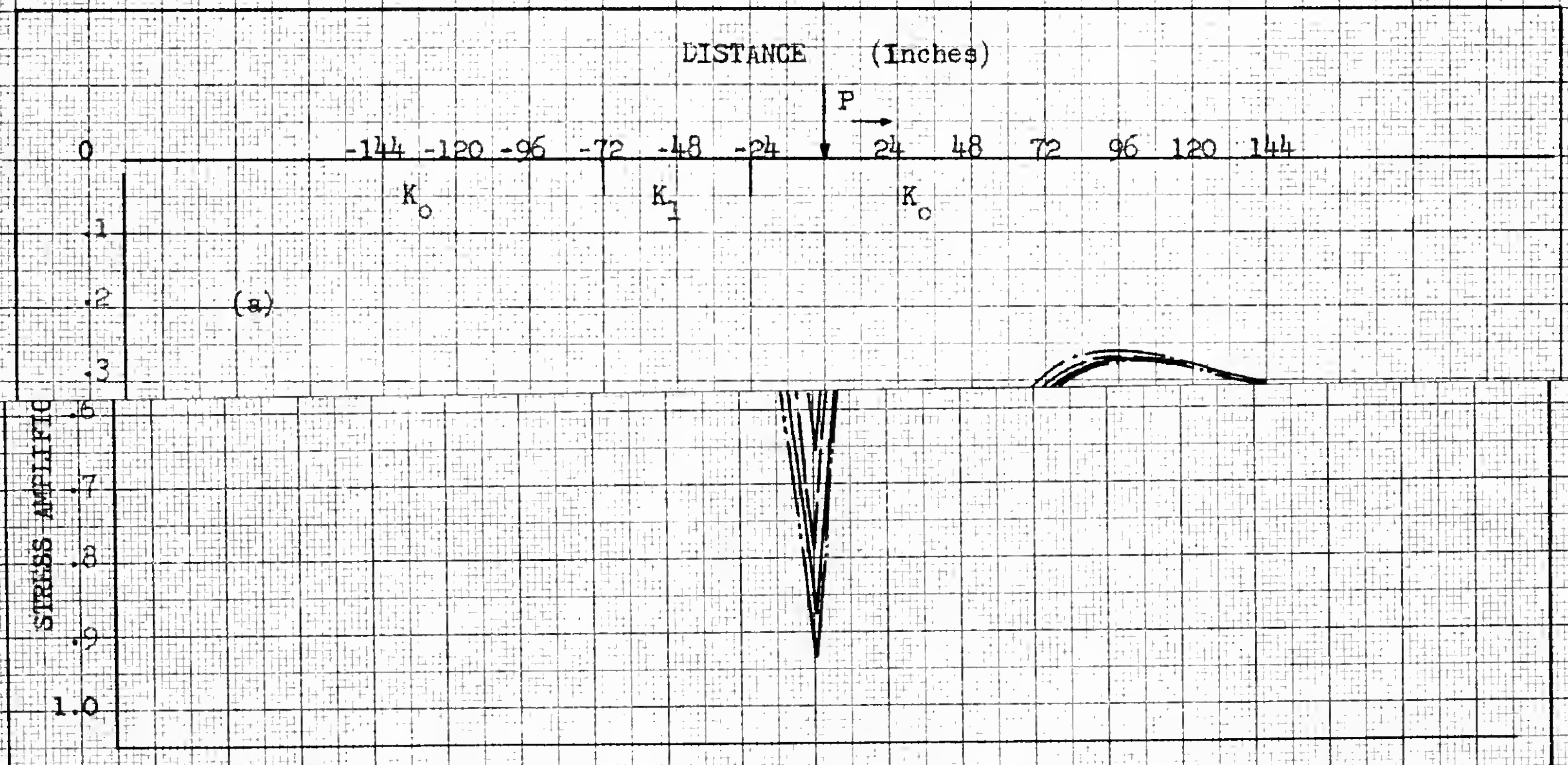


Figure 26. Deflection and Stress Amplification for 2 feet Ahead of Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

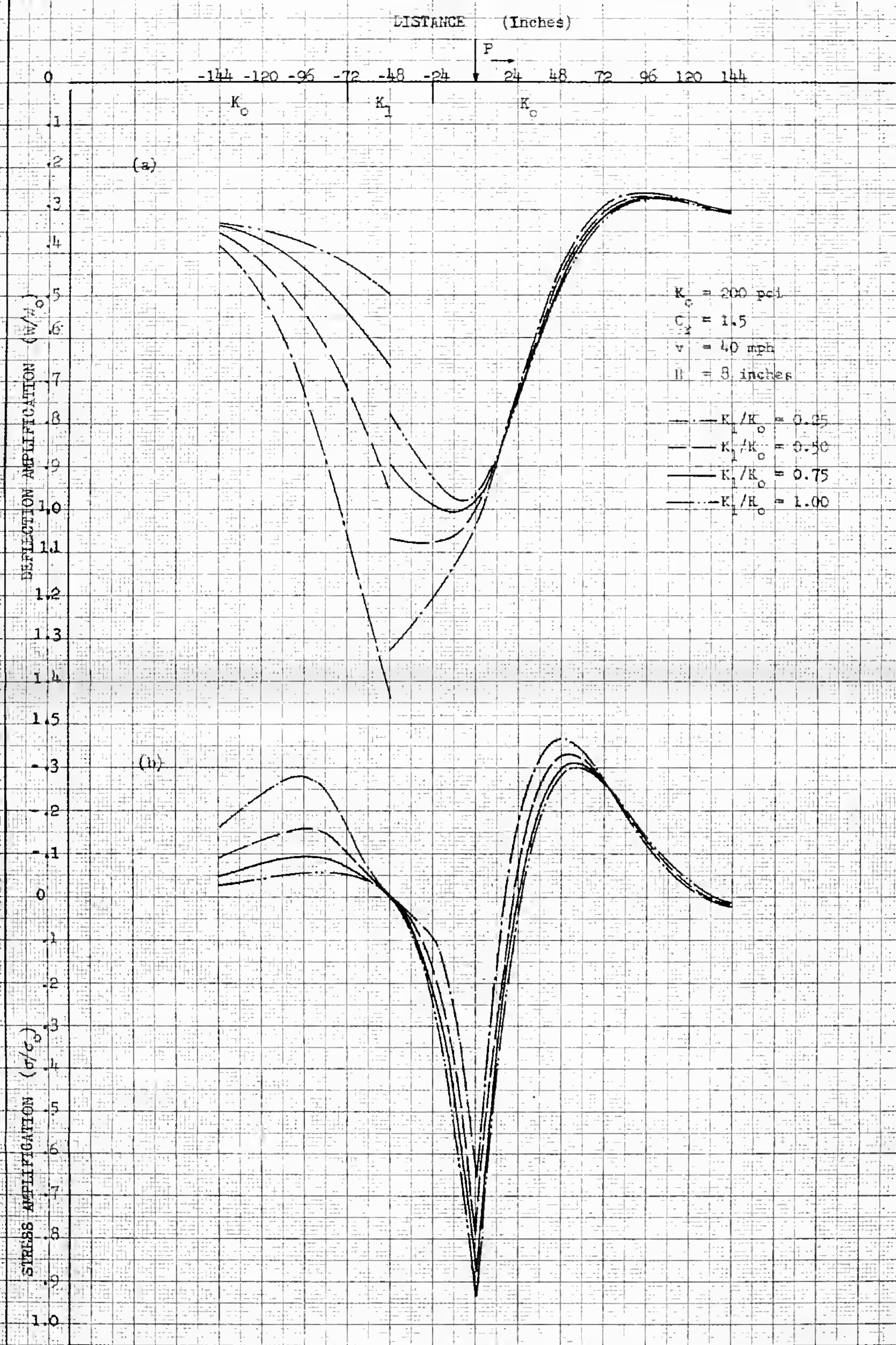


Figure 26. Deflection and Stress Amplification for 2 feet Ahead of Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)

## Discussion of Part II

As shown in Figures 11, 12 and 13, there is an increase in positive deflection amplification as the value of $K_1/K_o$ decreases. For the load positions shown, the increase is greatest when the load is over the zone of reduced subgrade reaction and least when the load is passed this zone. For example: for a value of $K_1/K_o = 0.5$, the load at mid-point (Figure 12) yields a deflection amplification ratio $\approx 1.28$, the load 2 ft. behind the zone of reduced-K (Figure 11) gives approximately 1.1 and the load 2 ft. in front of this zone yields $\approx 1.07$ (Figure 13).

As may be expected, when the load is moving over a homogeneous subgrade material, the point of maximum positive deflection lags the position of the load; however, if there is a zone of reduced subgrade reaction and the load is approaching this zone, as is shown in Figure 11, the maximum deflection can <u>lead</u> the position of the load at low values of $K_1/K_o$. For the cases when the load is over the weakened region or has passed it, maximum positive deflection <u>lags</u> the load. This characteristic tends to increase as the load moves out of the weakened area.

As regards to stresses, a reduction in the value of K does not appear to influence the maximum stress very much unless the load is within the area of reduced-K; however, as Figures 11 and 13 demonstrate, it can effect significantly the shape of the curve at very low values of $K_1/K_o$.

For the 3 positions of load considered, the maximum stress, which always occurred under the moving load, reached its greatest value when the load was over the weakened zone; here, stress increased as $K_1/K_o$ decreased. In the case when the load is approaching the weakened zone

stress is again seen to increase with decreasing ratio, $K_1/K_o$, but after the load has passed this zone, the trend is reversed and the lower the value of $K_1/K_o$, the lower the stress under the load. This means that if a slab is lying on a subgrade which exhibits a soft spot, the stress experienced by the slab is greater when the load is moving towards the center of this weakened zone than when it (the load) is moving away. Consequently, if failure due to overstressing does occur, the signs of distress should appear somewhere between the center and the "back" edge of the soft zone.

Figures 14 and 15 demonstrate that except for very low values of $K_1/K_o$, deflection and stress amplification decreases with increasing velocity, however the decrease appears to be more pronounced for the smaller value of initial subgrade reaction, $K_o$. Here, as in Part II, the higher value of $C_r$ yields the lower value of maximum deflection and stress.

The influence of the length of weakened region is illustrated in Figures 16, 17, 18, and 19. In these figures, both the maximum stress and deflection are seen to increase for all values of $K_1/K_o$ less than 1, as the region of reduced subgrade reaction becomes larger. The stiffer the subgrade material is initially, the greater the increase in deflection and stress. However, in the case of stress, this is only discernible at low values of $K_1/K_o$.

In general, the influence of the thickness of the pavement upon stress and deflection amplification was only significant at low ratios of $K_1/K_o$. As is shown in Figures 20 through 23, deflection amplification decreases as thickness increases, when the value of $K_1/K_o$ is approximately equal to 0.7 or less; while stress amplification remains relatively insensitive to variation in thickness. Again, these plots show that for

constant $K_1/K_o$ ratios less than 1, the initially stiffer subgrade leads to the greater increase in stress and deflection. This suggests that if there is the possibility for the development of soft spots in the subgrade material, it may be better to avoid the use of stiff subgrades.

In Figures 24, 25, and 26, there is a clear indication of what may happen, when, in addition to the reduction in subgrade reaction, 50% of load transfer is lost. Deflections are substantially increased near the point of discontinuity, expecially when the load is over the weakened region. For instance, in Figure 25a, for $K_1/K_o = 0.5$, the deflection amplification for a point under the moving load is 5.85 compared to 1.26 in Figure 12a. Another important aspect to note, is the large relative movement which occurs between the edges of the slab (i.e. edges situated at the mid-point of the weakened region). This movement not only causes bumpy driving, but may also lead to further pavement distress.

As for stresses, Figure 15b demonstrates that for the special case considered, the maximum stress need not occur under the moving load. In fact, the maximum stress experienced in this case is not only behind the load, but is also correspondingly higher than that obtained in Figure 12 . For the other two positions of the load (Figures 24b and 26b), the use of no-moment-transfer and 50% shear transfer did not greatly influence the values previously obtained in Figures 11 and 13 for maximum stress, as long as the ratio of $K_1/K_o$ was high (approximately = 0.8); however, when the ratio of $K_1/K_o$ was lowered to 0.25, significant reduction in stress was experienced. Obviously, at high values of $K_1/K_o$, the load in this case has to be fairly close to the weakened region before conditions existing at the mid-point of the region become important.

## SUMMARY

In Part I of this thesis, it is shown that when a pavement is subjected to upward curling, it is possible to experience an increase in maximum positive deflection as velocity is increased and then a decrease (in maximum positive deflection) with further increases of velocity. The velocity at which the decrease in maximum positive deflection sets in seems to depend on several factors which include the thickness of the slab, the temperature difference between its surfaces, the stiffness of the foundation and the degree of damping. In general, it appears that higher velocities are needed to effect a decrease in maximum positive deflection as slab thickness, stiffness of foundation and temperature increase, and as the degree of damping decreases.

In the case of stresses, maximum stress increased with increasing velocity. The thinner the pavement and the lower the value of subgrade reaction and damping coefficient, the higher the stress; however, there was not a great difference in the values of stress obtained. What seemed to matter most, was the difference in temperature between the surfaces of slab. Increases in the temperature difference resulted in large increases in stress and maximum positive deflection, for all values of velocity studied. This observation tends to indicate temperature difference between slab surfaces as the over-riding factor governing the magnitude of stress and deflection which may be obtained, while factors such as velocity, thickness of pavement, modulus of subgrade reaction and degree

of damping act more or less to exaggerate or minimize the pavement distress caused by temperature and/or moisture gradients.

In Part II, where the influence of a reduction in subgrade reaction was studied, it was clearly shown that both maximum deflection and maximum stress increased as the region became weaker. The increase experienced was quite pronounced when the load was within the weakened region and moving towards the center. Thus, if a pavement is designed with a particular value of subgrade reaction, and a weakened zone develops due to the softening of ground (as may be the case during Spring thaw) the stresses produced in the slab in the near vicinity of the weakened zone will very likely be higher than anticipated.

If there is significant reduction of subgrade support (even to within 75% of the original value of $K_o$), maximum deflection and stress should still follow the well known trend and decrease with increasing velocity; however, when there is complete loss of subgrade support, both the maximum stress and deflection can increase with increasing velocity. For the hypothetical case where pavements of equal thickness are built on different subgrade material, it appears that for a constant percentage loss of subgrade support, the pavement built on the stiffer subgrade material should experience a greater increase in stress and deflection. In the case where there is also a loss of load transfer, deflections as well as stresses may be substantially increased and large relative movement may be experienced near the points of discontinuity in the surface of the slab.

# CONCLUSIONS

1. On the basis of the assumptions stated herein, a theory has been developed whereby stresses and deflections can be computed for finite rectangular slabs subject to ambient and imposed moving loads. The base may exhibit time dependent effects and may also be non-homogeneous.

2. Contrary to common opinion, the theory demonstrated that an increase in velocity can produce an increase in deflection and stress.

3. Of all the variables considered herein (temperature differences, velocities, thicknesses of pavement, moduli of subgrade reaction and degrees of damping), the temperature difference between slab surfaces was shown to be the over-riding factor governing the magnitude of maximum stress and deflection.

4. There is reason to believe that any meaningful interpretation of the performance of pavements as determined by measured strains and/or deflections of the slab, must give due regard to the effects of warping.

5. The results of this study indicate that stresses and deflections within a slab can be sensitive to localized reductions in subgrade support.

## SUGGESTIONS FOR FURTHER RESEARCH

The procedure developed here could equally be applied to the case where the temperature at the top of the slab is greater than that at the bottom. Furthermore, other models such as the Maxwell and the Standard Solid may also be used to simulate support conditions.

At the present time, there is very little known on the range of values to be used in viscoelastic models and therefore a full scaled experiment is in order. Such an experiment should not only measure stresses and deflections, but should in particular pay attention to the prevailing boundary conditions. For example, some attempt should be made to evaluate the conditions at the joints in highway slabs.

The problem of a slab lying on a viscoelastic material and subject to warping as well as to a series of moving concentrated loads is also worthy of future study.

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# APPENDIX I

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## Nomenclature

$\Delta T$ = difference in temperature between the surfaces of the slab

$t$ = time

$w$ = mid-plane deflection of the slab (positive downward)

$x_1, y_1$ = fixed coordinates

$x, y$ = coordinates with respect to moving load

$E$ = Young's modulus

$\mu$ = Poisson's ratio

$H$ = slab thickness

$q_o$ = uniform distributed load due to the weight of the slab

$P$ = concentrated line load

$\rho$ = density of the slab

$K$ = modulus of subgrade reaction

$C$ = damping coefficient

$v$ = velocity of moving load

$D$ = $\dfrac{EH^3}{12(1-\mu^2)}$ = flexural rigidity of the slab

$a$ = length of slab

$\ell$ = length of region of reduced subgrade reaction

$L_1, L_2$ = distances from load to edges of region of reduced subgrade reaction

$b, c, d$ = distances from load to edges of zones in partially supported slab

$C_r$ = ratio of the damping coefficient to that of the critical damping coefficient

$V(x)$ = shear at any point x

$M(x)$ = bending moment at any point x

$\overline{V}(x)$ = shear at any point x in infinite slab

$\sigma(x)$ = tensile stress at any point x

$\beta$ = linear coefficient of thermal expansion

# APPENDIX II

# APPENDIX II

## GENERAL SOLUTION

### Part I

The general differential equation for the deflection of a thin rectangular plate resting on a viscoelastic foundation and subjected to a moving line load, P, is (42):

$$D \frac{\partial^4 w}{\partial x_1^4} + \rho H \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + Kw = q_o + P\,(x_1, t) \tag{1}$$

Employing the transformation $x = (x_1 - vt)$, and introducing the moving load, P, with the boundary conditions, Equation (1) becomes:

$$D \frac{d^4 w}{dx^4} + \rho H v^2 \frac{d^2 w}{dx^2} - Cv \frac{dw}{dx} + Kw = q_o \tag{2}$$

The solution of Equation (2) as given by Thompson (59) is:

For $\Delta > 0$

$$w = \frac{q_o}{K} + e^{-a_1 x} (N_1 \sin b_2 x + N_2 \cos b_2 x) + e^{a_1 x} (N_3 \sin b_1 x + N_4 \cos b_1 x) \tag{3a}$$

For $\Delta < 0$

$$w = \frac{q_o}{K} + N_5 e^{a_4 x} + N_6 e^{a_3 x} + e^{-a_1 x} (N_7 \sin b_2 x + N_8 \cos b_2 x) \tag{3b}$$

$$\text{where } a_1^6 + \frac{m^2}{2} a_1^4 + \left(\frac{m^4}{16} - \frac{K}{4D}\right) a_1^2 - \left(\frac{Cv}{8D}\right)^2 = 0 \tag{4a}$$

$$a_4 = a_1 + \sqrt{\frac{Cv}{4Da_1} - a_1^2 - \frac{m^2}{2}} \tag{4b}$$

$$a_3 = a_1 - \sqrt{\frac{Cv}{4Da_1} - a_1^2 - \frac{m^2}{2}} \tag{4c}$$

$$b_1 = \sqrt{a_1^2 - \frac{Cv}{4Da_1} + \frac{m^2}{2}} \tag{4d}$$

$$b_2 = \sqrt{a_1^2 + \frac{Cv}{4Da_1} + \frac{m^2}{2}} \tag{4e}$$

and $$m^2 = \frac{\rho Hv^2}{D} \tag{4f}$$

For the region where the plate is unsupported Equation (2) reduces to:

$$D \frac{d^4 w}{dx^4} + \rho Hv^2 \frac{d^2 w}{dx^2} = q_o \tag{5}$$

The solution to Equation (5) is:

$$w = N_9 + N_{10}x + N_{11} \cdot \cos(mx) + N_{12} \sin(mx) + \frac{q_o}{2D} \left(\frac{x}{m}\right)^2 \tag{6}$$

where m is given by Equation (4f).

To obtain the solution for the case described by Figure 1, the following conditions must be satisfied:

$$\begin{aligned}
&(a) \quad M_1(-b) = 0 \\
&(b) \quad V_1(-b) = \bar{V}(-b) \\
&(c) \quad w_1(-c) = 0 \\
&(d) \quad w_2(-c) = 0 \\
&(e) \quad w_1'(-c) = w_2'(-c) \\
&(f) \quad w_1''(-c) = w_2''(-c) \\
&(g) \quad w'''_1(-c) = w'''_2(-c) \\
&(h) \quad w_2(0) = w_3(0) \\
&(i) \quad w_2'(0) = w_3'(0) \\
&(j) \quad w_2''(0) = w_3''(0) \\
&(k) \quad w_3'''(0) - w_2'''(0) = P/D \\
&(l) \quad w_3(d) = 0 \\
&(m) \quad w_4(d) = 0
\end{aligned} \tag{7}$$

(n) $w'_3(d) = w'_4(d)$

(o) $w''_3(d) = w''_4(d)$

(p) $w'''_3(d) = w'''_4(d)$

(q) $M_4\ (a-b) = 0$

(r) $V_4\ (a-b) = \bar{V}\ (a-b)$

For the case where $\Delta > 0$, the deflection of the plate may be expressed as follows:

$$w_1 = A_1 + A_2 x + A_3 \cos (mx) + A_4 \sin (mx) + \frac{q_o}{2D} \left(\frac{x}{m}\right)^2 \tag{8}$$

$$w_2 = \frac{q_o}{K} + e^{-a_1 x} (B_1 \sin b_2 x + B_2 \cos b_2 x) + e^{a_1 x} (B_3 \sin b_1 x + B_4 \cos b_1 x) \tag{9}$$

$$w_3 = \frac{q_o}{K} + e^{-a_1 x} (C_1 \sin b_2 x + C_2 \cos b_2 x) + e^{a_1 x} (C_3 \sin b_1 x + C_4 \cos b_1 x) \tag{10}$$

$$w_4 = D_1 + D_2 x + D_3 \cos (mx) + D_4 \sin (mx) + \frac{q_o}{2D} \left(\frac{x}{m}\right)^2 \tag{11}$$

The bending moment in a thin plate subject to a gradient $\Delta T/H$ (61) is:

$$M = -D \left[ \frac{d^2 w}{dx^2} + \beta (1+\mu) \frac{\Delta T}{H} \right] \tag{12}$$

For $M_1(-b) = 0$, Equation (7a)

$$A_3 m^2 \cos (mb) - A_4 m^2 \sin (mb) = \frac{q_o}{m^2 D} + \beta(1+\mu) \frac{\Delta T}{H} \tag{13}$$

The shear in a thin plate subject to a gradient $\Delta T/H$ (61) is:

$$V = -D \frac{d}{dx} \left[ \frac{d^2 w}{dx^2} + \beta (1+\mu) \frac{\Delta T}{H} \right] \tag{14}$$

As the gradient is assumed to be independent of x, Equation (14) reduces to:

$$V = - D \left(\frac{d^3w}{dx^3}\right) \tag{15}$$

For $V_1(-b) = \overline{V}(-b)$, Equation (7b)

$$A_3m^3 \sin (mb) + A_4m^3 \cos (mb) = \frac{\overline{V}(-b)}{D} \tag{16}$$

Solving Equations (13) and (16)

$$A_3 = \frac{1}{m^2}\left[\frac{\overline{V}(-b)\sin(mb)}{mD} + \left(\frac{q_o}{m^2D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\cos(mb)\right] \tag{17a}$$

$$A_4 = \frac{1}{m^2}\left[\frac{\overline{V}(-b)\cos(mb)}{mD} - \left(\frac{q_o}{m^2D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\sin(mb)\right] \tag{17b}$$

For $M_4(a-b) = 0$, Equation (7q)

$$D_3m^2 \cos m(a-b) + D_4m^2 \sin m(a-b) = \frac{q_o}{m^2D} + \beta(1+\mu)\frac{\Delta T}{H} \tag{18}$$

For $V_4(a-b) = \overline{V}(a-b)$, Equation (7r)

$$-D_3m^3 \sin m(a-b) + D_4m^3 \cos m(a-b) = \frac{\overline{V}(a-b)}{D} \tag{19}$$

Solving Equations (18) and (19)

$$D_3 = -\frac{1}{m^2}\left[\frac{\overline{V}(a-b)}{mD}\sin m(a-b) - \left(\frac{q_o}{m^2D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\cos m(a-b)\right] \tag{20a}$$

$$D_4 = \frac{1}{m^2}\left[\frac{\overline{V}(a-b)}{mD}\cos m(a-b) + \left(\frac{q_o}{m^2D} + \beta(1+\mu)\frac{\Delta T}{H}\right)\sin m(a-b)\right] \tag{20b}$$

Applying Equations (7c) through (7p) to Equations (8), (9), (10) and (11), the following set of non-linear equations are obtained.

$$A_1 - A_2c + A_3 \cos(mc) - A_4 \sin(mc) = -\frac{q_o}{2D}\left(\frac{c}{m}\right)^2 \tag{21}$$

$$e^{a_1c}(B_2 \cos b_2c - B_1 \sin b_2c) + e^{-a_1c}(B_4 \cos b_1c - B_3 \sin b_1c) =$$

$$- q_o/K \tag{22}$$

$$e^{a_1c}\left[B_1(b_2 \cos b_2c + a_1 \sin b_2c) - B_2(a_1 \cos b_2c - b_2 \sin b_2c)\right]$$

$$+ e^{-a_1c}\left[B_3(b_1 \cos b_1c - a_1 \sin b_1c) + B_4(b_1 \sin b_1c + a_1 \cos b_1c)\right]$$

$$- A_2 - A_3m \sin mc - A_4m \cos mc = -\frac{q_oc}{m^2D} \tag{23}$$

$$e^{a_1c}\left[B_2\left[(a_1^2 - b_2^2) \cos b_2c - 2a_1b_2 \sin b_2c\right] - B_1\left[(a_1^2 - b_2^2) \sin b_2c\right.\right.$$

$$\left.\left. + 2a_1b_2 \cos b_2c\right]\right] + e^{-a_1c}\left[B_4\left[(a_1^2 - b_1^2) \cos b_1c + 2a_1b_1 \sin b_1c\right]\right.$$

$$\left. + B_3\left[2a_1b_1 \cos b_1c - (a_1^2 - b_1^2) \sin b_1c\right]\right] + m^2(A_3 \cos mc - A_4 \sin mc)$$

$$= \frac{q_o}{m^2D} \tag{24}$$

$$e^{a_1 c}\Big[B_2\left[(3a_1b_2^2 - a_1^3)\cos b_2c - (b_2^3 - 3a_1^2b_2)\sin b_2c\right] +$$

$$B_1\left[(3a_1^2b_2 - b_2^3)\cos b_2c - (3a_1b_2^2 - a_1^3)\sin b_2c\right]\Big] -$$

$$e^{-a_1 c}\Big[B_3\left[(a_1^3 - 3a_1b_1^2)\sin b_1c + (b_1^3 - 3a_1^2b_1)\cos b_1c\right] +$$

$$B_4\left[(b_1^3 - 3a_1^2b_1)\sin b_1c - (a_1^3 - 3a_1b_1^2)\cos b_1c\right]\Big] + .$$

$$m^3 (A_3 \sin mc + A_4 \cos mc) = 0 \tag{25}$$

$$B_2 + B_4 - C_2 - C_4 = 0 \tag{26}$$

$$b_2(B_1 - C_1) + a_1(C_2 - C_4 - B_2 + B_4) + b_1(B_3 - C_3) = 0 \tag{27}$$

$$2a_1b_2(C_1 - B_1) + (a_1^2 - b_2^2)(B_2 - C_2) + 2a_1b_1(B_3 - C_3) +$$

$$(a_1^2 - b_1^2)(B_4 - C_4) = 0 \tag{28}$$

$$(b_2^3 - 3b_2a_1^2)(B_1 - C_1) - (3a_1b_2^2 - a_1^3)(B_2 - C_2) +$$

$$(b_1^3 - 3b_1a_1^2)(B_3 - C_3) + (3a_1b_1^2 - a_1^3)(B_4 - C_4) = \frac{P}{D} \tag{29}$$

$$e^{-a_1 d}(C_1 \sin b_2d + C_2 \cos b_2d) + e^{a_1 d}(C_3 \sin b_1d +$$

$$C_4 \cos b_1d) = -q_o/K \tag{30}$$

$$D_1 + D_2 d + D_3 \cos md + D_4 \sin md = -\frac{q_o}{2D}\left(\frac{d}{m}\right)^2 \tag{31}$$

$$e^{-a_1 d}\left[C_1(b_2 \cos b_2 d - a_1 \sin b_2 d) - C_2(a_1 \cos b_2 d + b_2 \sin b_2 d)\right] +$$

$$e^{a_1 d}\left[C_3(a_1 \sin b_1 d + b_1 \cos b_1 d) + C_4(a_1 \cos b_1 d - b_1 \sin b_1 d)\right] -$$

$$D_2 + m\,(D_3 \sin md - D_4 \cos md) = \frac{q_o d}{m^2 D} \tag{32}$$

$$e^{-a_1 d}\Big[C_1\left[(a_1^2 - b_2^2) \sin b_2 d - 2a_1 b_2 \cos b_2 d\right] + C_2\left[2a_1 b_2 \sin b_2 d\right.$$

$$\left. + (a_1^2 - b_2^2) \cos b_2 d\right]\Big] + e^{a_1 d}\Big[C_3\left[(a_1^2 - b_1^2) \sin b_1 d + 2a_1 b_1 \cos b_1 d\right]$$

$$+ C_4\left[(a_1^2 - b_1^2) \cos b_1 d - 2a_1 b_1 \sin b_1 d\right]\Big] + m^2(D_3 \cos md + D_4 \sin md)$$

$$= \frac{q_o}{m^2 D} \tag{33}$$

$$e^{-a_1 d}\Big[C_1\left[(3a_1 b_2^2 - a_1^3) \sin b_2 d - (b_2^3 - 3b_2 a_1^2) \cos b_2 d\right] +$$

$$C_2\left[(3a_1 b_2^2 - a_1^3) \cos b_2 d + (b_2^3 - 3b_2 a_1^2) \sin b_2 d\right]\Big] +$$

$$e^{a_1 d}\Big[C_4\left[(b_1^3 - 3b_1 a_1^2) \sin b_1 d - (3a_1 b_1^2 - a_1^3) \cos b_1 d\right] -$$

$$C_3\left[(3a_1 b_1^2 - a_1^3) \sin b_1 d + (b_1^3 - 3b_1 a_1^2) \cos b_1 d\right]\Big] -$$

$$m^3(D_3 \sin md - D_4 \cos md) = 0 \tag{34}$$

With the use of N-dimensional Newton's Method (27), the constants in Equations 21 through 34 may be determined and the deflections obtained from Equations 8, 9, 10 and 11.

Using Equation 12 to express the bending moment, the stress throughout the thin plate is given by $\sigma = 6M/H^2$.

For $-b \leq x \leq -c$

$$\sigma_1 = \frac{6D}{H^2}\left[m^2(A_3 \cos mx + A_4 \sin mx) - \frac{q_o}{m^2 D} - \beta(1+\mu)\frac{\Delta T}{H}\right] \tag{35}$$

For $-c \leq x \leq 0$

$$\begin{aligned}\sigma_2 = \frac{-6D}{H^2}\Big[ & e^{-a_1x}\Big(B_1\left[(a_1^2 - b_2^2)\sin b_2x - 2a_1b_2 \cos b_2x\right] + \\ & B_2\left[(a_1^2 - b_2^2)\cos b_2x + 2a_1b_2 \sin b_2x\right]\Big) + \\ & e^{a_1x}\Big(B_3\left[(a_1^2 - b_1^2)\sin b_1x + 2a_1b_1 \cos b_1x\right] + \\ & B_4\left[(a_1^2 - b_1^2)\cos b_1x - 2a_1b_1 \sin b_1x\right]\Big) + \beta(1+\mu)\frac{\Delta T}{H}\Big]\end{aligned} \tag{36}$$

For $0 \leq x \leq d$

$$\begin{aligned}\sigma_3 = \frac{-6D}{H^2}\Big[ & e^{-a_1x}\Big(C_1\left[(a_1^2 - b_2^2)\sin b_2x - 2a_1b_2 \cos b_2x\right] + \\ & C_2\left[(a_1^2 - b_2^2)\cos b_2x + 2a_1b_2 \sin b_2x\right]\Big) + \\ & e^{a_1x}\Big(C_3\left[(a_1^2 - b_1^2)\sin b_1x + 2a_1b_1 \cos b_1x\right] + \\ & C_4\left[(a_1^2 - b_1^2)\cos b_1x - 2a_1b_1 \sin b_1x\right]\Big) + \beta(1+\mu)\frac{\Delta T}{H}\Big]\end{aligned} \tag{37}$$

For $d \leq x \leq (a-b)$

$$\sigma_4 = \frac{6D}{H^2} \left[ m^2 (D_3 \cos mx + D_4 \sin mx) - \frac{q_o}{m^2 D} - \beta (1+\mu) \frac{\Delta T}{H} \right] \tag{38}$$

For the case where $\Delta < 0$, the deflection of the plate in zones 1 and 4 (Figure 1) is given by Equations 8 and 11 respectively. In zones 2 and 3, the deflection may be expressed as follows:

$$w_2 = q_o/K + B_5 e^{a_4 x} + B_6 e^{a_3 x} + e^{-a_1 x} (B_7 \sin b_2 x + B_8 \cos b_2 x) \tag{39}$$

$$w_3 = q_o/K + C_5 e^{a_4 x} + C_6 e^{a_3 x} + e^{-a_3 x} (C_7 \sin b_2 x + C_8 \cos b_2 x) \tag{40}$$

Using the same boundary conditions, Equation 7, and following the procedure previously outlined, the constants may be evaluated and the stresses and deflections determined.

To obtain the complete solution to the problem, the method of solution is applied to the various regions which develop (for example, see Figure 2).

## Part II

The solution to the governing differential equation (Equation 1) is again given by Equation 3, therefore, for the case where $\Delta_1$, $\Delta_2$, and $\Delta_3 > 0$, and the problem is again described by Figure 9b, ($L_1 \geq 0$) the deflection of the plate may be expressed as follows:

$$w_{1a} = q_o/K_o + e^{a_1 x} (A_5 \sin b_1 x + A_6 \cos b_1 x) + e^{-a_1 x} (A_7 \sin b_2 x + A_8 \cos b_2 x) \tag{41}$$

$$w_{1b} = q_o/K_o + e^{a_1x}(B_5 \sin b_1x + B_6 \cos b_1x) + e^{-a_1x}(B_7 \sin b_2x + B_8 \cos b_2x) \tag{42}$$

$$w_2 = q_o/K_1 + e^{a_2x}(C_5 \sin \gamma_1x + C_6 \cos \gamma_1x) + e^{-a_2x}(C_7 \sin \gamma_2x + C_8 \cos \gamma_2x) \tag{43}$$

$$w_3 = q_o/K_o + e^{a_1x}(D_5 \sin b_1x + D_6 \cos b_1x) + e^{-a_1x}(D_7 \sin b_2x + D_8 \cos b_2x) \tag{44}$$

To obtain the particular solution of interest, the following conditions must be satisfied:

$$\begin{aligned}
&(a)\quad w_{1a}(-\infty) = q_o/K_o \\
&(b)\quad w'_{1a}(-\infty) = 0 \\
&(c)\quad w_{1a}(0) = w_{1b}(0) \\
&(d)\quad w'_{1a}(0) = w'_{1b}(0) \\
&(e)\quad w''_{1a}(0) = w''_{1b}(0) \\
&(f)\quad w'''_{1b}(0) - w'''_{1a}(0) = P/D \\
&(g)\quad w_{1b}(L_1) = w_2(L_1) \\
&(h)\quad w'_{1b}(L_1) = w'_2(L_1) \\
&(i)\quad w''_{1b}(L_1) = w''_2(L_1) \\
&(j)\quad w'''_{1b}(L_1) = w'''_2(L_1) \\
&(k)\quad w_2(L_2) = w_3(L_2) \\
&(l)\quad w'_2(L_2) = w'_3(L_2) \\
&(m)\quad w''_2(L_2) = w''_3(L_2) \\
&(n)\quad w'''_2(L_2) = w'''_3(L_2)
\end{aligned} \tag{45}$$

(o) $w_3(\infty) = q_o/K_o$

(p) $w_3'(\infty) = 0$

As x approaches $-\infty$, both $w_{1a}$ and $w_{1a}'$ approach $\infty$, therefore if conditions (a) and (b) are to be satisfied, then $A_7$ and $A_8$ must equal zero. Similarly, conditions (o) and (p) imply that $D_5$ and $D_6$ must equal zero.

Applying conditions (c) through (n) to Equations 41 through 44, the following equations are obtained:

$$A_6 - B_6 - B_8 = 0 \tag{46}$$

$$b_1A_5 + a_1A_6 - b_1B_5 - a_1B_6 - b_2B_7 + a_1B_8 = 0 \tag{47}$$

$$2a_1b_1A_5 + (a_1^2 - b_1^2)\,A_6 - 2a_1b_1B_5 - (a_1^2 - b_1^2)\,B_6 + 2a_1b_2B_7 - (a_1^2 - b_2^2)\,B_8 = 0 \tag{48}$$

$$(b_1^3 - 3b_1a_1^2)\,A_5 - (a_1^3 - 3a_1b_1^2)\,A_6 - (b_1^3 - 3b_1a_1^2)\,B_5 + (a_1^3 - 3a_1b_1^2)\,B_6 + (3a_1^2b_2 - b_2^3)\,B_7 + (3a_1b_2^2 - a_1^3)\,B_8 = P/D \tag{49}$$

$$e^{a_1L_1}(B_5 \sin b_1L_1 + B_6 \cos b_1L_1) + e^{-a_1L_1}(B_7 \sin b_2L_1 + B_8 \cos b_2L_1) - e^{a_2L_1}(C_5 \sin \gamma_1L_1 + C_6 \cos \gamma_1L_1) - e^{-a_2L_1}(C_7 \sin \gamma_2L_1 + C_8 \cos \gamma_2L_1) = q_o\left(\frac{K_o - K_1}{K_1\,K_o}\right) \tag{50}$$

$$e^{a_1L_1}\left[B_5(b_1 \cos b_1L_1 + a_1 \sin b_1L_1) + B_6(a_1 \cos b_1L_1 - b_1 \sin b_1L_1)\right]$$

$$+ e^{-a_1L_1}\left[B_7(b_2 \cos b_2L_1 - a_1 \sin b_2L_1) - B_8(b_2 \sin b_2L_1 + a_1 \cos b_2L_1)\right]$$

$$- e^{a_2L_1}\left[C_5(\gamma_1 \cos \gamma_1L_1 + a_2 \sin \gamma_1L_1) + C_6(a_2 \cos \gamma_1L_1 - \gamma_1 \sin \gamma_1L_1)\right]$$

$$- e^{-a_2L_1}\left[C_7(\gamma_2 \cos \gamma_2L_1 - a_2 \sin \gamma_2L_1) - C_8(\gamma_2 \sin \gamma_2L_1 + a_2 \cos \gamma_2L_1)\right]$$

$$= 0 \tag{51}$$

$$e^{a_1L_1}\Big[B_5\left[(a_1^2 - b_1^2) \sin b_1L_1 + 2a_1b_1 \cos b_1L_1\right] + B_6\left[(a_1^2 - b_1^2) \cos b_1L_1\right.$$

$$\left.- 2a_1b_1 \sin b_1L_1\right]\Big] + e^{-a_1L_1}\Big[B_7\left[(a_1^2 - b_2^2) \sin b_2L_1 - 2a_1b_2 \cos b_2L_1\right]$$

$$+ B_8\left[2a_1b_2 \sin b_2L_1 + (a_1^2 - b_2^2) \cos b_2L_1\right]\Big]$$

$$-e^{a_2L_1}\Big[C_5\left[(a_2^2 - \gamma_1^2) \sin \gamma_1L_1 + 2a_2\gamma_1 \cos \gamma_1L_1\right] + C_6\left[(a_2^2 - \gamma_1^2) \cos \gamma_1L_1\right.$$

$$\left.- 2a_2\gamma_1 \sin \gamma_1L_1\right]\Big] - e^{-a_2L_1}\Big[C_7\left[(a_2^2 - \gamma_2^2) \sin \gamma_2L_1 - 2a_2\gamma_2 \cos \gamma_2L_1\right]$$

$$+ C_8\left[2a_2\gamma_2 \sin \gamma_2L_1 + (a_2^2 - \gamma_2^2) \cos \gamma_2L_1\right]\Big] = 0 \tag{52}$$

$$e^{a_1L_1}\Big[B_5[(a_1^3 - 3a_1b_1^2)\sin b_1L_1 - (b_1^3 - 3b_1a_1^2)\cos b_1L_1]+$$

$$B_6[(b_1^3 - 3b_1a_1^2)\sin b_1L_1 + (a_1^3 - 3a_1b_1^2)\cos b_1L_1]\Big]+$$

$$e^{-a_1L_1}\Big[B_7[(3a_1^2b_2 - b_2^3)\cos b_2L_1 + (3a_1b_2^2 - a_1^3)\sin b_2L_1]+$$

$$B_8[(3a_1b_2^2 - a_1^3)\cos b_2L_1 + (b_2^3 - 3a_1^2b_2)\sin b_2L_1]\Big]$$

$$- e^{a_2L_1}\Big[C_5[(a_2^3 - 3a_2\gamma_1^2)\sin \gamma_1L_1 - (\gamma_1^3 - 3\gamma_1a_2^2)\cos \gamma_1L_1]+$$

$$C_6[(\gamma_1^3 - 3\gamma_1a_2^2)\sin \gamma_1L_1 + (a_2^3 - 3a_2\gamma_1^2)\cos \gamma_1L_1]\Big] -$$

$$e^{-a_2L_1}\Big[C_7[(3a_2^2\gamma_2 - \gamma_2^3)\cos \gamma_2L_1 + (3a_2\gamma_2^2 - a_2^3)\sin \gamma_2L_1]+$$

$$C_8[(3a_2\gamma_2^2 - a_2^3)\cos \gamma_2L_1 + (\gamma_2^3 - 3a_2^2\gamma_2)\sin \gamma_2L_1]\Big] = 0 \tag{53}$$

$$e^{a_2L_2}(C_5 \sin \gamma_1L_2 + C_6 \cos \gamma_1L_2) + e^{-a_2L_2}(C_7 \sin \gamma_2L_2 + C_8 \cos \gamma_2L_2)$$

$$- e^{-a_1L_2}(D_7 \sin b_2L_2 + D_8 \cos b_2L_2) = q_o\left(\frac{K_1 - K_o}{K_1\ K_o}\right) \tag{54}$$

$$e^{a_2L_2}[C_5(\gamma_1 \cos \gamma_1L_2 + a_2 \sin \gamma_1L_2) + C_6(a_2 \cos \gamma_1L_2 - \gamma_1 \sin \gamma_1L_2)]$$

$$+ e^{-a_2L_2}[C_7(\gamma_2 \cos \gamma_2L_2 - a_2 \sin \gamma_2L_2) - C_8(\gamma_2 \sin \gamma_2L_2 + a_2 \cos \gamma_2L_2)] -$$

$$e^{-a_1L_2}[D_7(b_2 \cos b_2L_2 - a_1 \sin b_2L_2) - D_8(b_2 \sin b_2L_2 + a_1 \cos b_2L_2)]$$

$$= 0 \tag{55}$$

$$e^{a_2L_2}\Big[C_5\,[(a_2^2 - \gamma_1^2)\sin\gamma_1L_2 + 2a_2\gamma_1\cos\gamma_1L_2] +$$

$$C_6\,[(a_2^2 - \gamma_1^2)\cos\gamma_1L_2 - 2a_2\gamma_1\sin\gamma_1L_2]\Big] + e^{-a_2L_2}\Big[C_7\,[(a_2^2 - \gamma_2^2)\sin\gamma_2L_2$$

$$- 2a_2\gamma_2\cos\gamma_2L_2] + C_8\,[(a_2^2 - \gamma_2^2)\cos\gamma_2L_2 + 2a_2\gamma_2\sin\gamma_2L_2]\Big] -$$

$$e^{-a_1L_2}\Big[D_7\,[(a_1^2 - b_2^2)\sin b_2L_2 - 2a_1b_2\cos b_2L_2] +$$

$$D_8\,[(a_1^2 - b_2^2)\cos b_2L_2 + 2a_1b_2\sin b_2L_2]\Big] = 0 \qquad (56)$$

$$e^{a_2L_2}\Big[C_5\,[(a_2^3 - 3a_2\gamma_1^2)\sin\gamma_1L_2 - (\gamma_1^3 - 3\gamma_1a_2^2)\cos\gamma_1L_2] +$$

$$C_6\,[(\gamma_1^3 - 3\gamma_1a_2^2)\sin\gamma_1L_2 + (a_2^3 - 3a_2\gamma_1^2)\cos\gamma_1L_2]\Big] +$$

$$e^{-a_2L_2}\Big[C_7\,[(3a_2^2\gamma_2 - \gamma_2^3)\cos\gamma_2L_2 + (3a_2\gamma_2^2 - a_2^3)\sin\gamma_2L_2] +$$

$$C_8\,[(3a_2\gamma_2^2 - a_2^3)\cos\gamma_2L_2 + (\gamma_2^3 - 3a_2^2\gamma_2)\sin\gamma_2L_2]\Big] -$$

$$e^{-a_1L_2}\Big[D_7\,[(3a_1^2b_2 - b_2^3)\cos b_2L_2 + (3a_1b_2^2 - a_1^3)\sin b_2L_2] +$$

$$D_8\,[(3a_1b_2^2 - a_1^3)\cos b_2L_2 + (b_2^3 - 3a_1^2b_2)\sin b_2L_2]\Big] = 0 \qquad (57)$$

The constants in Equations 46 through 57 were evaluated with the use of a 7090 computer, and the deflections of the plate were determined from Equations (41), (42), (43) and (44).

The bending moment (M) of the plate is given by Equation 12, ($\Delta T = 0$), and the stress (per inch width) is expressed by $\sigma = 6M/H^2$. For the case described by Figure 9b, the stresses throughout the various zones are as follows:

For $x \le 0$

$$\sigma_{1a} = \frac{-6De^{a_1x}}{H^2}\Big[ A_5\left[(a_1^2 - b_1^2)\sin b_1x + 2a_1b_1\cos b_1x\right] +$$

$$A_6\left[(a_1^2 - b_1^2)\cos b_1x - 2a_1b_1\sin b_1x\right]\Big] \tag{58}$$

For $0 \le x \le L_1$

$$\sigma_{1b} = \frac{-6D}{H^2}\Big[ e^{a_1x}\Big(B_5\left[(a_1^2 - b_1^2)\sin b_1x + 2a_1b_1\cos b_1x\right] +$$

$$B_6\left[(a_1^2 - b_1^2)\cos b_1x - 2a_1b_1\sin b_1x\right]\Big) +$$

$$e^{-a_1x}\Big(B_7\left[(a_1^2 - b_2^2)\sin b_2x - 2a_1b_2\cos b_2x\right] +$$

$$B_8\left[(a_1^2 - b_2^2)\cos b_2x + 2a_1b_2\sin b_2x\right]\Big)\Big] \tag{59}$$

For $L_1 \le x \le L_2$

$$\sigma_2 = \frac{-6D}{H^2}\Big[ e^{a_2x}\Big(C_5\left[(a_2^2 - \gamma_1^2)\sin \gamma_1x + 2a_2\gamma_1\cos \gamma_1x\right] +$$

$$C_6\left[(a_2^2 - \gamma_1^2)\cos \gamma_1x - 2a_2\gamma_1\sin \gamma_1x\right]\Big) +$$

$$e^{-a_2x}\Big(C_7\left[(a_2^2 - \gamma_2^2)\sin \gamma_2x - 2a_2\gamma_2\cos \gamma_2x\right] +$$

$$C_8\left[2a_2\gamma_2\sin \gamma_2x + (a_2^2 - \gamma_2^2)\cos \gamma_2x\right]\Big)\Big] \tag{60}$$

For $x \geq L_2$

$$\sigma_3 = \frac{-6D}{H^2}\left[ e^{-a_1 x} \left( D_7 \left[ (a_1^2 - b_2^2) \sin b_2 x - 2a_1 b_2 \cos b_2 x \right] + \right.\right.$$

$$\left.\left. D_8 \left[ (a_1^2 - b_2^2) \cos b_2 x + 2a_1 b_2 \sin b_2 x \right] \right) \right] \tag{61}$$

For the condition when $0 \geq L_1 \geq -\ell$ (see Figure 9c) the deflection of the plate becomes:

$$w_1 = {}^{q}o/K_o + e^{a_1 x}(A_5 \sin b_1 x + A_6 \cos b_1 x) + e^{-a_1 x}(A_7 \sin b_2 x +$$

$$A_8 \cos b_2 x) \tag{62}$$

$$w_{2a} = {}^{q}o/K_1 + e^{a_2 x}(C_5 \sin \gamma_1 x + C_6 \cos \gamma_1 x) + e^{-a_2 x}(C_7 \sin \gamma_2 x +$$

$$C_8 \cos \gamma_2 x) \tag{63}$$

$$w_{2b} = {}^{q}o/K_1 + e^{a_2 x}(C_9 \sin \gamma_1 x + C_{10} \cos \gamma_1 x) + e^{-a_2 x}(C_{11} \sin \gamma_2 x +$$

$$C_{12} \cos \gamma_2 x) \tag{64}$$

$$w_3 = {}^{q}o/K_o + e^{a_1 x}(D_5 \sin b_1 x + D_6 \cos b_1 x) + e^{-a_1 x}(D_7 \sin b_2 x +$$

$$D_8 \cos b_2 x) \tag{65}$$

and the boundary conditions which must be satisfied are:

$$\begin{aligned}
&(a)\quad w_1 \ (-\infty) = {}^{q}o/K_o \\
&(b)\quad w_1' \ (-\infty) = 0 \\
&(c)\quad w_1(L_1) = w_{2a}(L_1) \\
&(d)\quad w_1'(L_1) = w_{2a}'(L_1) \\
&(e)\quad w_1''(L_1) = w_{2a}''(L_1) \\
&(f)\quad w_1'''(L_1) = w_{2a}'''(L_1)
\end{aligned} \tag{66}$$

(g) $w_{2a}(0) = w_{2b}(0)$

(h) $w'_{2a}(0) = w'_{2b}(0)$

(i) $w''_{2a}(0) = w''_{2b}(0)$

(j) $w'''_{2b}(0) - w'''_{2a}(0) = P/D$

(k) $w_{2b}(L_2) = w_3(L_2)$

(l) $w'_{2b}(L_2) = w'_3(L_2)$

(m) $w''_{2b}(L_2) = w''_3(L_2)$

(n) $w'''_{2b}(L_2) = w'''_3(L_2)$

(o) $w_3(\infty) = q_\sigma/K_o$

(p) $w'_3(\infty) = 0$

Once again, the deflections and stresses may be evaluated by applying the boundary conditions (Equation 66) to Equations 62, 63, 64 and 65 and solving for the constants in the resulting set of linear equations.

To solve the problem when $L_2 < 0$ (Figure 9d), the deflection of the plate is expressed as follows:

$$w_1 = q_o/K_o + e^{a_1 x}(A_5 \sin b_1 x + A_6 \cos b_1 x) + e^{-a_1 x}(A_7 \sin b_2 x + A_8 \cos b_2 x) \tag{67}$$

$$w_2 = q_o/K_1 + e^{a_2 x}(C_5 \sin \gamma_1 x + C_6 \cos \gamma_1 x) + e^{-a_2 x}(C_7 \sin \gamma_2 x + C_8 \cos \gamma_2 x) \tag{68}$$

$$w_{3a} = q_o/K_o + e^{a_1 x}(D_5 \sin b_1 x + D_6 \cos b_1 x) + e^{-a_1 x}(D_7 \sin b_2 x + D_8 \cos b_2 x) \tag{69}$$

$$w_{3b} = q_o/K_o + e^{a_1 x}(D_9 \sin b_1 x + D_{10} \cos b_1 x) + e^{-a_1 x}(D_{11} \sin b_2 x + D_{12} \cos b_2 x) \tag{70}$$

and the following conditions are used to evaluate the constants:

$$
\begin{aligned}
&\text{(a)} \quad w_1(-\infty) = q_o/K_o \\
&\text{(b)} \quad w_1'(-\infty) = 0 \\
&\text{(c)} \quad w_1(L_1) = w_2(L_1) \\
&\text{(d)} \quad w_1'(L_1) = w_2'(L_1) \\
&\text{(e)} \quad w_1''(L_1) = w_2''(L_1) \\
&\text{(f)} \quad w_1'''(L_1) = w_2'''(L_1) \\
&\text{(g)} \quad w_2(L_2) = w_{3a}(L_2) \\
&\text{(h)} \quad w_2'(L_2) = w_{3a}'(L_2) \\
&\text{(i)} \quad w_2''(L_2) = w_{3a}''(L_2) \\
&\text{(j)} \quad w_2'''(L_2) = w_{3a}'''(L_2) \\
&\text{(k)} \quad w_{3a}(0) = w_{3b}(0) \\
&\text{(l)} \quad w_{3a}'(0) = w_{3b}'(0) \\
&\text{(m)} \quad w_{3a}''(0) = w_{3b}''(0) \\
&\text{(n)} \quad w_{3b}'''(0) - w_{3a}'''(0) = P/D \\
&\text{(o)} \quad w_{3b}(\infty) = q_o/K_o \\
&\text{(p)} \quad w_{3b}'(\infty) = 0
\end{aligned} \tag{71}
$$

For the case where the value of $\Delta$ becomes less than zero for a particular zone, the procedure to be followed and the boundary conditions are the same as previously described, but the appropriate solution to the governing differential equation must be used for the respective zones (see Equations 3 and 6).

# APPENDIX III

## Typical Program for Part I (see Figure 1)

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C      SOL. TO NON-LINEAR PROBLEM
C      DELI IS GREATER THAN ZERO
       DIMENSION A(14,15), INDEX(14)
       DIMENSION AK(1), CR(1), BT(1), VR(1)
       DIMENSION W(42), AMOM(42), SIGMA(42)
       DIMENSION X(14), AA(1), BA(1), BN(1)
       COMMON X
       COMMON AA,BA,BN,E,GM,HM,GNM,HNM,GN1,HN1,QN1,FN1,F1,G1,H1,Q1,
      1B11,B12,D11,D12,QV,QK,V,P,D,TEM
 5001  FORMAT (6F10.2)
 5020  I=1
       READ (5,5001) AK(I), CR(I), HH, DH, BH, TE
       AL=480.
       EY=4000000.
       RHO=.27100000E-02
       P=125.
       AMU=0.15
       BE=0.000006
       F=1000000.
       WRITE (6,5001) AK(1), CR(1), HH, DH, BH, AL,TE
       WRITE (6,1002)
 3001  FORMAT (7E17.8)
 1002  FORMAT (/)
       DO 90 KI=1,3
       HCOUNT=KI
       H=HH+2.*(HCOUNT-1.)
       QQ=H*(150.9/1728.)
       QK=QQ/AK(1)
       D=EY*H**3/(12.*(1.-AMU**2))
       TEM=-BE*TE*(1.+AMU)/H
       MKK=3
       IF( TE .EQ. 20. .AND. CR(1) .EQ. 1.5) MKK=4
       IF ( TE .GT. 25. .AND. CR(1) .EQ. 1.5) MKK=5
       IF ( TE .GT. 25. .AND. CR(1) .EQ. 2.0) MKK=4
       DO 91 KK=2,MKK
       VCOUNT=KK
       V=352.*(VCOUNT-1.)
       E=RHO*H*V**2/D
       QV=QQ/(V**2*RHO*H)
       VR(I)=V/(SQRT(SQRT(4.*AK(I)*D/((RHO*H)**2))))
       DELI  =16.*(4.*(1.-CR(I)**2)*VR(I)**8-(8.-36.*CR(I)**2+
      127.*CR(I)**4)*VR(I)**4+4.)
       BT(I)=SQRT(SQRT(AK(I)/(4.*D)))
       B1=F
       B2=2.*VR(I)*VR(I)*BT(I)*BT(I)*F
       B3=(BT(I)**4)*((VR(I)**4)-1.)*F
       B4=(VR(I)**2)*(CR(I)**2)*F*(BT(I)**6)
       EPS=.10000000E-04
       XX=0.02
       CALL SITER (XX,EPS,B1,B2,B3,B4)
       AA(I)=ABS(XX)
       BN(I)=BT(I)*SQRT(2.*VR(I)**2+(AA(I)/BT(I))**2
      1+(2.*VR(I)*CR(I)*BT(I))/AA(I))
       WRITE (6,3001) H,V,VR(1),BT(1),AA(1),BN(1),DELI
       GN1=BN(1)**3-3.*BN(1)*AA(1)**2
```

```
      HN1=3.*AA(1)*BN(1)**2-AA(1)**3
      QN1=2.*AA(1)*BN(1)
      FN1=AA(1)**2-BN(1)**2
      GNM=GN1*BN(1)-AA(1)*HN1
      HNM=BN(1)*HN1+AA(1)*GN1
      IF (DEL1-0.) 83,83,82
83    GO TC 91
82    BA(I)=BT(I)*SQRT(2.*VR(I)**2+(AA(I)/BT(I))**2
     1-(2.*VR(I)*CR(I)*BT(I))/AA(I))
      F1=AA(1)**2-BA(1)**2
      G1=BA(1)**3-3.*BA(1)*AA(1)**2
      H1=3.*AA(1)*BA(1)**2-AA(1)**3
      Q1=2.*AA(1)*BA(1)
      GM=G1*BA(1)-AA(1)*H1
      HM=BA(1)*H1+AA(1)*G1
      C4=P*AA(1)/(D*(AA(1)**2*(4.*AA(1)**2+BN(1)**2+3.*BA(1)**2)+
     1(F1-FN1)*(4.*AA(1)**2-FN1+F1)/4.))
      C3=C4*(-4.*AA(1)**2+FN1-F1)/(2.*Q1)
      C1=C4*(4.*AA(1)**2+FN1-F1)/(2.*QN1)
      MMK=1
      IF (TE .EQ. 20. .AND. V .EQ. 704.) MMK=2
      IF (TE .EQ. 20. .AND. V .EC. 704. .AND. CR(1) .EQ. 1.5 .AND.
     1 H .EQ. 12.) MMK=1
      IF ( TE .EQ. 20. .AND. V .EQ. 1056.) MMK=3
      IF ( TE .EQ. 30. .AND. V .EQ. 1056. .ANC. CR(1) .EQ. 1.5) MMK=2
      IF ( TE .EQ. 30. .AND. V .EQ. 1056. .ANC. CR(1) .EQ. 2.0) MMK=3
      IF ( TE .EQ. 30. .AND. V .EQ. 1408.) MMK=5
      IF ( TE .EQ. 30. .AND. V .EQ. 1408. .ANC. H .EQ. 8.) MMK=6
      IF ( TE .EQ. 40. .AND. V .EQ. 1408.) MMK=3
      IF ( TE .EQ. 40. .AND. V .EQ. 1056. .ANC. CR(1) .EQ. 2.0) MMK=2
      DO 211 II=MMK,9
      SCCUNT=II
      SS1=-(48.+(SCOUNT-1.)*DH)
      SS3=AL+SS1
      AML=0.
      AVL=EXP(AA(1)*SS1)*(C4*(G1*SIN(BA(1)*SS1)-H1*COS(BA(1)*SS1))-
     1C3*(H1*SIN(BA(1)*SS1)+G1*COS(BA(1)*SS1)))
      AVR=EXP(-AA(1)*SS3)*(C4*(HN1*COS(BN(1)*SS3)+GN1*SIN(BN(1)*SS3))+
     1C1*(HN1*SIN(BN(1)*SS3)-GN1*COS(BN(1)*SS3)))
      AMR=0.
      B11=QQ*COS(SQRT(E)*SS3)/(E**2*D)-((TEM+AMR)*COS(SQRT(E)*SS3)-
     1AVR*SIN(SQRT(E)*SS3)/SQRT(E))/E
      B12=CQ*SIN(SQRT(E)*SS3)/(E**2*D)-((TEM+AMR)*SIN(SQRT(E)*SS3)+
     1AVR*COS(SQRT(E)*SS3)/SQRT(E))/E
      D11=QQ*COS(SQRT(E)*SS1)/(E**2*D)-((TEM+AML)*COS(SQRT(E)*SS1)-
     1AVL*SIN(SQRT(E)*SS1)/SQRT(E))/E
      D12=CQ*SIN(SQRT(E)*SS1)/(E**2*D)-((TEM+AML)*SIN(SQRT(E)*SS1)+
     1AVL*COS(SQRT(E)*SS1)/SQRT(E))/E
      QP=10000.
      DO 412 J=1,12
      DO 412 K=1,13
412   A(J,K)=0.
      X(13)=SS1
      SZ=SIN(BA(1)*X(13))
      RZ=COS(BA(1)*X(13))
      TZ=EXP(AA(1)*X(13))
```

APPENDIX III

```
      DZ=EXP(-AA(1)*X(13))
      SNZ=SIN(BN(1)*X(13))
      RNZ=COS(BN(1)*X(13))
      RMZ=COS(SQRT(E)*X(13))
      SMZ=SIN(SQRT(E)*X(13))
      A(1,1)=TZ*(F1*RZ-Q1*SZ)
      A(1,2)=TZ*(Q1*RZ+F1*SZ)
      A(1,3)=DZ*(FN1*RNZ+QN1*SNZ)
      A(1,4)=DZ*(FN1*SNZ-QN1*RNZ)
      A(2,1)=TZ*(G1*SZ-H1*RZ)
      A(2,2)=-TZ*(H1*SZ+G1*RZ)
      A(2,3)=DZ*(HN1*RNZ+GN1*SNZ)
      A(2,4)=DZ*(HN1*SNZ-GN1*RNZ)
      A(3,1)=1.
      A(3,3)=1.
      A(4,1)=AA(1)
      A(4,2)=BA(1)
      A(4,3)=-AA(1)
      A(4,4)=BN(1)
      A(5,1)=F1
      A(5,2)=Q1
      A(5,3)=FN1
      A(5,4)=-QN1
      A(6,1)=H1
      A(6,2)=G1
      A(6,3)=-HN1
      A(6,4)=GN1
      DO 413 J=3,6
      DO 413 K=1,4
      JJ=K+4
  413 A(J,JJ)=-A(J,K)
      X(14)=SS3-12.*VCOUNT
      IF (SS1 .EQ. -432.) X(14)=12.
      SX=SIN(BA(1)*X(14))
      RX=COS(BA(1)*X(14))
      TX=EXP(AA(1)*X(14))
      DX=EXP(-AA(1)*X(14))
      SNX=SIN(BN(1)*X(14))
      RNX=COS(BN(1)*X(14))
      RMX=COS(SQRT(E)*X(14))
      SMX=SIN(SQRT(E)*X(14))
      A(7,5)=TX*(AA(1)*RX-BA(1)*SX)
      A(7,6)=TX*(AA(1)*SX+BA(1)*RX)
      A(7,7)=-DX*(AA(1)*RNX+BN(1)*SNX)
      A(7,8)=DX*(BN(1)*RNX-AA(1)*SNX)
      A(7,10)=-1.
      A(8,5)=TX*(F1*RX-Q1*SX)
      A(8,6)=TX*(F1*SX+Q1*RX)
      A(8,7)=DX*(QN1*SNX+FN1*RNX)
      A(8,8)=DX*(FN1*SNX-QN1*RNX)
      A(9,5)=TX*(G1*SX-H1*RX)
      A(9,6)=-TX*(H1*SX+G1*RX)
      A(9,7)=DX*(HN1*RNX+GN1*SNX)
      A(9,8)=DX*(HN1*SNX-GN1*RNX)
      A(10,5)=TX*RX
      A(10,6)=TX*SX
```

```
      A(10,7)=DX*RNX
      A(10,8)=DX*SNX
      A(10,9)=-1.
      A(10,10)=-X(14)
      A(11,1)=TZ*RZ
      A(11,2)=TZ*SZ
      A(11,3)=DZ*RNZ
      A(11,4)=DZ*SNZ
      A(11,11)=-1.
      A(11,12)=-X(13)
      A(12,1)=TZ*(AA(1)*RZ-BA(1)*SZ)
      A(12,2)=TZ*(AA(1)*SZ+BA(1)*RZ)
      A(12,3)=-DZ*(AA(1)*RNZ+BN(1)*SNZ)
      A(12,4)=DZ*(BN(1)*RNZ-AA(1)*SNZ)
      A(12,12)=-1.
      A(1,13)=QV-E*(D11*RMZ+D12*SMZ)
      A(2,13)=SQRT(E**3)*(D11*SMZ-D12*RMZ)
      A(6,13)=P/D
      A(7,13)=SQRT(E)*(B12*RMX-B11*SMX)+QV*X(14)
      A(8,13)=QV-E*(B11*RMX+B12*SMX)
      A(9,13)=SQRT(E**3)*(B11*SMX-B12*RMX)
      A(10,13)=B11*RMX+B12*SMX+QV*X(14)**2/2.-QK
      A(11,13)=D11*RMZ+D12*SMZ+QV*X(13)**2/2.-QK
      A(12,13)=SQRT(E)*(D12*RMZ-D11*SMZ)+QV*X(13)
      DO 213 J=1,12
      DO 213 K=1,13
  213 A(J,K)=QP*A(J,K)
      CALL CROUT (A,12,1,13,DETERM,INDEX)
      DO 214 J=1,12
      K=1
  214 X(J)=A(J,K)
      WRITE (6,3001) (X(J), J=1,14)
      WRITE (6,1002)
      N=14
      EPS=0.1
      ISW=1
      CALL NONLIN (N,X,EPS,ISW)
      WRITE (6,1002)
      IF (X(14) .LT. 481. .AND. X(14) .GT. -0.5) GO TO 400
      WRITE (6,3001) SS1, X(14), H, V, CR(1)
      GO TO 211
  400 DO 101 JJJ=1,41
      XCOUNT=JJJ
      Y=SS1+(XCOUNT-1.)*BH
      SY=SIN(BA(1)*Y)
      RY=COS(BA(1)*Y)
      SNY=SIN(BN(1)*Y)
      RNY=COS(BN(1)*Y)
      IF (Y-X(13)) 150, 150, 151
  150 W(JJJ)=X(11)+X(12)*Y+D11*COS(SQRT(E)*Y)+D12*SIN(SQRT(E)*Y)+
     1QV*Y**2/2.
      AMOM(JJJ)=-D*E*(D11*(COS(SQRT(E)*SS1)-COS(SQRT(E)*Y))+
     1D12*(SIN(SQRT(E)*SS1)-SIN(SQRT(E)*Y))+AML/E)
      SIGMA(JJJ)=6.*AMOM(JJJ)/(H**2)
      GO TO 101
  151 IF (Y-0.) 152,152,153
```

```
152  W(JJJ)=CK+EXP(-AA(1)*Y)*(X(4)*SNY+X(3)*RNY)+
    1EXP(AA(1)*Y)*(X(2)*SY+X(1)*RY)
     AMCM(JJJ)=-D*(EXP(-AA(1)*Y)*(X(4)*(FN1*SNY-QN1*RNY)+
    1X(3)*(QN1*SNY+FN1*RNY))+EXP(AA(1)*Y)*(X(2)*(F1*SY+Q1*RY)+
    2X(1)*(F1*RY-C1*SY))-TEM)
     SIGMA(JJJ)=6.*AMOM(JJJ)/(H**2)
     GO TO 101
153  IF (Y-X(14)) 154,154,155
154  W(JJJ)=QK+EXP(-AA(1)*Y)*(X(8)*SNY+X(7)*RNY)+
    1EXP(AA(1)*Y)*(X(6)*SY+X(5)*RY)
     AMCM(JJJ)=-D*(EXP(-AA(1)*Y)*(X(8)*(FN1*SNY-QN1*RNY)+
    1X(7)*(QN1*SNY+FN1*RNY))+EXP(AA(1)*Y)*(X(6)*(F1*SY+Q1*RY)+
    2X(5)*(F1*RY-Q1*SY))-TEM)
     SIGMA(JJJ)=6.*AMOM(JJJ)/(H**2)
     GO TO 101
155  W(JJJ)=X(9)+X(10)*Y+B11*COS(SQRT(E)*Y)+B12*SIN(SQRT(E)*Y)+
    1QV*Y**2/2.
     AMCM(JJJ)=-D*E*(B11*(COS(SQRT(E)*SS3)-CCS(SQRT(E)*Y))+
    1B12*(SIN(SQRT(E)*SS3)-SIN(SQRT(E)*Y))+AMR/E)
     SIGMA(JJJ)=6.*AMOM(JJJ)/(H**2)
 101 CCNTINUE
     WRITE (6,3001) (W(JJJ), JJJ=1,41)
     WRITE (6,1002)
     WRITE (6,3001) (SIGMA(JJJ), JJJ=1,41)
     WRITE (6,3001) SS1
     WRITE (6,1002)
 211 CONTINUE
 91  CCNTINUE
 90  CCNTINUE
     GO TC 5020
     END
```

```
     SUBROUTINE SITER (XX,EPS,B1,B2,B3,B4)
 24  XNB=XX
     XN1=B1*XNB**6+B2*XNB**4+B3*XNB**2-B4
     XN2=6.*B1*XNB**5+4.*B2*XNB**3+2.*B3*XNB
     XX=XNB-XN1/XN2
     IF (ABS(XX-XNB)-EPS) 26,26,25
 25  GO TO 24
 26  RETURN
     END
```

```
      SUBROUTINE NONLIN (N,X,EPS,ISW)
C     N IS NUMBER OF INDEPENDENT VARIABLES ANC EQUATIONS
C      X IS INITIAL ESTIMATE OF ROOT AND MUST HAVE DIMENSION N IN
C        CALLING PROGRAM
C     EPS IS ALLOWED ABSOLUTE ERROR
C     ISW IS INTERMEDIATE OUTPUT SELECTOR
C         1 FOR NO INTERMEDIATE OUTPUT
C         2 FOR INTERMEDIATE OUTPUT
      DIMENSION F(14,14), G(14), DELT(14), X(14), REX(14), S(14),
     1BEST(14), A(14,15), INDEX(14)
      COMMON X
      REPS = .01*EPS
      DO 1 I=1,14
      G(I)=0.0
      DO 1 J=1,14
    1 F(I,J)=0.0
      SSAEX = 1.E6
      SSAET = 0.0
      IC=0
    2 ISC=1
      K=0
    3 CALL EVAL(F,G)
C     X IS IN COMMON
      DO 17 I=1,N
   17 G(I) = -G(I)
      DO 30 I = 1,N
   30 S(I) = G(I)
      DM = 1.0
      NN=N+1
      QP=1000.
      DO 987 I=1,N
      DO 987 J=1,N
  987 A(I,J)=QP*F(I,J)
      DO 988 I=1,N
  988 A(I,NN)=QP*G(I)
      CALL CROUT (A,N,1,NN,DETERM,INDEX)
      DO 100 IJK=1,N
 100  DELT (IJK)=A(IJK,1)
      IF(ABS(DETERM)-1.E-20)4,4,10
    4 K=K+1
      GO TO (5,6,7),K
    5 X(ISC)=1.1* X(ISC)
      GO TO 3
    6 X(ISC)= .9*X(ISC)/1.1
      GO TO 3
    7 X(ISC)= 1.1*X(ISC)
      ISC=ISC + 1
      IF(ISC-N) 8,8,9
    8 K=0
      GO TO 4
    9 L=1
      CALL PUNT(L, BEST, IC, ICB, N, ITRPT)
C     ZILCH
   10 DO 21 I=1,N
   21 REX(I) = ABS(DELT(I)/X(I))
```

```
      SSAET = 0.0
      DO 23 I=1,N
   23 SSAET = SSAET + S(I)**2
      IF(SSAET -SSAEX) 24,24,26
   24 DO 27 I = 1,N
   27 BEST(I) = X(I)
      SSAEX = SSAET
      ICB = IC + 1
   26 DO 11 I=1,N
      IF(REX(I)-REPS) 11,11,12
   11 CONTINUE
      DO 25 I = 1,N
      IF (S(I)-EPS) 25,25,12
   25 CONTINUE
      GO TO 15
   12 IC=IC+1
      IF(ISW-1)14,13,14
   13 IF(IC-100)18,18,19
   14 CALL INTER(N,IC,REX,S)
      IF(IC-100)18,18,19
   18 DO 20 I=1,N
   20 X(I) =X(I)+DELT(I)
      IF (X(14) .LT. -24. .OR. X(14) .GT. 500.) RETURN
      DIMENSION CYCLE(20,5)
      JC = MOD((IC-1),5)+1
      DO 40 I = 1,N
   40 CYCLE(I,JC) = X(I)
      IF (IC-5) 42,41,41
   42 K = IC
      GO TO 43
   41 K = 5
   43 DO 50 J = 1,K
      IF (J-JC) 44,50,44
   44 DO 46 I = 1,N
      IF (CYCLE(I,JC) -CYCLE(I,J))50 ,46,50
   46 CONTINUE
      ITRPT = IC-MOD((5+JC-J),5)
      L = 3
      CALL PUNT(L, BEST, IC, ICB, N, ITRPT)
   50 CONTINUE
      GO TO 2
   15 SSREX =0.0
      DO 16 I=1,N
      X(I)=X(I)+DELT(I)
   16 SSREX =SSREX +REX(I)**2
      CALL FINAL(N,SSREX,SSAEX)
      RETURN
   19 L=2
      CALL PUNT(L, BEST, IC, ICB, N, ITRPT)
      RETURN
      END
```

```
      SUBROUTINE PUNT(L,BEST,IC,ICB,N,ITRPT)
      DIMENSION X(14), BEST(14)
      COMMON X
      GO TO (1,2,7),L
    1 ICT = IC+1
      WRITE (6,3) ICT
    3 FORMAT(1H 31H SYSTEM IS IN A SINGULAR REGION/1H 35H SINGULARITY OC
     1CURRED ON ITERATION I4/1H 27H THE SINGULAR POINT FOLLOWS)
      WRITE (6,4) (X(I),I=1,N)
    4 FORMAT(1H E20.8)
      STOP
    2 WRITE (6,5) ICB
    5 FORMAT(1H 33H NUMBER OF ITERATIONS EXCEEDS 100/ 1H 11H ITERATION
     1I4, 44H IS BEST ESTIMATE SO FAR AND IS GIVEN BELOW )
      WRITE (6,6) (BEST(I),I=1,N)
    6 FORMAT(1H E20.8)
      STOP
    7 WRITE (6,10) ITRPT,IC
   10 FORMAT(13H1 ITERATIONS I3, 4H AND I3,46H ARE IDENTICAL INDICATING
     1 A CYCLIC CONDITION. /43H   THE BEST RESULTS SO FAR ARE GIVEN BELO
     2W.  )
      WRITE (6,6) (BEST(I),I=1,N)
      STOP
      END
```

```
      SUBROUTINE INTER(N,IC,REX,S)
      DIMENSION X(14), REX(14), S(14)
      COMMON X
      WRITE (6,1) IC
    1 FORMAT(1HO 20H ITERATION COUNT IS I3)
      WRITE (6,2)
    2 FORMAT(1H 5X, 55H ESTIMATED ROOT      RELATIVE ERROR      ABSOLUTE
     1 ERROR)
      WRITE (6,3)(X(I),REX(I),S(I),I=1,N)
    3 FORMAT(1H 3E20.8)
      RETURN
      END
```

```
      SUBROUTINE FINAL(N,SSREX,SSAEX)
      DIMENSION X(14)
      COMMON X
      WRITE (6,1)
    1 FORMAT(1HO 20X, 12H FINAL ROOT )
      WRITE (6,2) (X(I),I=1,N)
    2 FORMAT(1H 20X, E20.8)
      WRITE (6,3) SSREX
    3 FORMAT(1H 10X, 38H SUM OF SQUARES OF RELATIVE ERRORS IS E20.9 )
      WRITE (6,4) SSAEX
    4 FORMAT(1H 10X, 38H SUM OF SQUARES OF ABSOLUTE ERRORS IS E20.9)
      RETURN
      END
```

```
C       CRCUT REDUCTION
        SUBROUTINE CROUT (A,N,M,NN,DETERM,INDEX)
        DIMENSION A(14,15), INDEX(14)
        DET=1.0
        JZ=N-1
        JA=N+1
        DO 30 I=1,N
   30   INDEX(I)=I
        DO 700 J=1,NN
        DO 800 II=1,N
        SUM=0.0
        I=INDEX(II)
        IF(II-J)33,34,34
   33   IF(II-1) 9000,9200,9000
 9000   LLLL=II-1
        DO 9100 K=1,LLLL
        IPPP=INDEX(K)
 9100   SUM=SUM+A(I,K)*A(IPPP,J)
 9200   A(I,J)=(A(I,J)-SUM)/A(I,II)
        GO TO 800
   34   IF(J-1) 8000,8200,8000
 8000   LLLL=J-1
        DO 8100 K=1,LLLL
        IPPP=INDEX(K)
 8100   SUM=SUM+A(I,K)*A(IPPP,J)
 8200   A(I,J)=A(I,J)-SUM
  800   CONTINUE
        IF(J-N)41,700,700
   41   L=INDEX(J)
        KA=L
        HIGH=A(L,J)
        KZ=0
        DO 35 I=J,JZ
        JC=I+1
        L=INDEX(JC)
        IF(ABS (HIGH)-ABS (A(L,J)))36,35,35
   36   HIGH=A(L,J)
        KA=L
        KZ=1
   35   CONTINUE
        IF(KZ.NE. 0)  DET=-DET
        IF(ABS (HIGH)-1.E-05)31,31,3200
   31   WRITE(6,32) HIGH
   32   FORMAT(48HOTHE PIVOT ELEMENT IS LESS THAN 1.E-05 VALUE IS E20.8)
 3200   DO 37 K=1,N
        KK=K
        IF(INDEX(K)-KA)37,38,37
   37   CONTINUE
   38   ITEMP=INDEX(J)
        INDEX(J)=INDEX(KK)
        INDEX(KK)=ITEMP
  700   CONTINUE
        IF(M)2000,1000,2000
 2000   L=N-1
        DO 39 J=JA,NN
```

```
      LL=1
      DO 42 K=1,N
      IF(ABS (A(K,J))-0.0)43,42,43
   42 CONTINUE
      IZ=INDEX(N)
      IF(ABS (A(IZ,N))-1.0E-02)46,46,44
   44 WRITE(6,45)
   45 FORMAT(59H0                                  ONLY SOLUTION IS ZERO VEC
     1TOR)
      GO TO 10
   46 A(IZ,J)=5.0000
      IZZ=INDEX(N-1)
      IF(ABS (A(IZZ,N))-1.0E-04)47,47,43
   47 A(IZZ,J)=2.50000
      LL=2
   43 DO 40 IJ=LL,L
      SUM1=0.0
      II=N-IJ
      I=INDEX(II)
      LL=II+1
      DO 9300 K=LL,N
      IP=INDEX(K)
 9300 SUM1=SUM1+A(I,K)*A(IP,J)
      A(I,J)=A(I,J)-SUM1
   40 CONTINUE
   39 CONTINUE
 1000 DETERM=1.0
      DO 900 I=1,N
      K=INDEX(I)
  900 DETERM=DETERM*A(K,I)
      DETERM=DETERM*DET
      DO 400 I=1,N
      DO 400 J=JA,NN
      K=INDEX(I)
      L=J-N
  400 A(I,L)=A(K,J)
   10 RETURN
      END
```

```
      SUBROUTINE EVAL(F,G)
      DIMENSION F(14,14), G(14), AA(1), BA(1), BN(1), X(14)
      COMMON X
      COMMON AA,BA,BN,E,GM,HM,GNM,HNM,GN1,HN1,QN1,FN1,F1,G1,H1,Q1,
     1B11,B12,D11,D12,QV,QK,V,P,D,TEM
      DO 512 J=1,14
      DO 512 K=1,14
  512 F(J,K)=0.
      RMZ=COS(SQRT(E)*X(13))
      SMZ=SIN(SQRT(E)*X(13))
      SZ=SIN(BA(1)*X(13))
      RZ=COS(BA(1)*X(13))
      TZ=EXP(AA(1)*X(13))
      DZ=EXP(-AA(1)*X(13))
      SNZ=SIN(BN(1)*X(13))
      RNZ=COS(BN(1)*X(13))
      F(1,1)=TZ*(F1*RZ-Q1*SZ)
      F(1,2)=TZ*(Q1*RZ+F1*SZ)
      F(1,3)=DZ*(FN1*RNZ+QN1*SNZ)
      F(1,4)=DZ*(FN1*SNZ-QN1*RNZ)
      F(2,1)=TZ*(G1*SZ-H1*RZ)
      F(2,2)=-TZ*(H1*SZ+G1*RZ)
      F(2,3)=DZ*(HN1*RNZ+GN1*SNZ)
      F(2,4)=DZ*(HN1*SNZ-GN1*RNZ)
      F(2,13)=TZ*(X(1)*(GM*RZ+HM*SZ)-X(2)*(HM*RZ-GM*SZ))+
     1DZ*(X(3)*(GNM*RNZ-HNM*SNZ)+X(4)*(HNM*RNZ+GNM*SNZ))-
     2E**2*(D11*RMZ+D12*SMZ)
      F(3,1)=1.
      F(3,3)=1.
      F(4,1)=AA(1)
      F(4,2)=BA(1)
      F(4,3)=-AA(1)
      F(4,4)=BN(1)
      F(5,1)=F1
      F(5,2)=Q1
      F(5,3)=FN1
      F(5,4)=-QN1
      F(6,1)=H1
      F(6,2)=G1
      F(6,3)=-HN1
      F(6,4)=GN1
      DO 513 J=3,6
      DO 513 K=1,4
      JJ=K+4
  513 F(J,JJ)=-F(J,K)
      SX=SIN(BA(1)*X(14))
      RX=COS(BA(1)*X(14))
      TX=EXP(AA(1)*X(14))
      DX=EXP(-AA(1)*X(14))
      SNX=SIN(BN(1)*X(14))
      RNX=COS(BN(1)*X(14))
      SMX=SIN(SQRT(E)*X(14))
      RMX=COS(SQRT(E)*X(14))
      F(7,5)=TX*(AA(1)*RX-BA(1)*SX)
      F(7,6)=TX*(AA(1)*SX+BA(1)*RX)
```

```
 F(7,7)=-DX*(AA(1)*RNX+BN(1)*SNX)
 F(7,8)=DX*(BN(1)*RNX-AA(1)*SNX)
 F(7,10)=-1.
 F(8,5)=TX*(F1*RX-Q1*SX)
 F(8,6)=TX*(F1*SX+Q1*RX)
 F(8,7)=DX*(QN1*SNX+FN1*RNX)
 F(8,8)=DX*(FN1*SNX-QN1*RNX)
 F(9,5)=TX*(G1*SX-H1*RX)
 F(9,6)=-TX*(H1*SX+G1*RX)
 F(9,7)=DX*(HN1*RNX+GN1*SNX)
 F(9,8)=DX*(HN1*SNX-GN1*RNX)
 F(9,14)=TX*(X(5)*(GM*RX+HM*SX)-X(6)*(HM*RX-GM*SX))+
1DX*(X(7)*(GNM*RNX-HNM*SNX)+X(8)*(HNM*RNX+GNM*SNX))-
2E**2*(B11*RMX+B12*SMX)
 F(10,5)=TX*RX
 F(10,6)=TX*SX
 F(10,7)=DX*RNX
 F(10,8)=DX*SNX
 F(10,14)=F(7,5)*X(5)+F(7,6)*X(6)+F(7,7)*X(7)+F(7,8)*X(8)
 F(11,1)=TZ*RZ
 F(11,2)=TZ*SZ
 F(11,3)=DZ*RNZ
 F(11,4)=DZ*SNZ
 F(12,1)=TZ*(AA(1)*RZ-BA(1)*SZ)
 F(12,2)=TZ*(AA(1)*SZ+BA(1)*RZ)
 F(12,3)=-DZ*(AA(1)*RNZ+BN(1)*SNZ)
 F(12,4)=DZ*(BN(1)*RNZ-AA(1)*SNZ)
 F(11,13)=F(12,1)*X(1)+F(12,2)*X(2)+F(12,3)*X(3)+F(12,4)*X(4)
 F(12,12)=-1.
 F(13,9)=1.
 F(13,10)=X(14)
 F(13,14)=X(10)-SQRT(E)*(B11*SMX-B12*RMX)+QV*X(14)
 F(14,11)=1.
 F(14,12)=X(13)
 F(14,13)=X(12)-SQRT(E)*(C11*SMZ-D12*RMZ)+QV*X(13)
 G(1)=F(1,1)*X(1)+F(1,2)*X(2)+F(1,3)*X(3)+F(1,4)*X(4)+E*(D11*RMZ+
1D12*SMZ)-QV
 G(2)=F(2,1)*X(1)+F(2,2)*X(2)+F(2,3)*X(3)+F(2,4)*X(4)-
1SQRT(E**3)*(C11*SMZ-D12*RMZ)
 G(3)=X(1)+X(3)-X(5)-X(7)
 G(4)=AA(1)*X(1)+BA(1)*X(2)-AA(1)*X(3)+BN(1)*X(4)-AA(1)*X(5)-
1BA(1)*X(6)+AA(1)*X(7)-BN(1)*X(8)
 G(5)=F1*X(1)+Q1*X(2)+FN1*X(3)-QN1*X(4)-F1*X(5)-Q1*X(6)-
1FN1*X(7)+QN1*X(8)
 G(6)=H1*X(1)+G1*X(2)-HN1*X(3)+GN1*X(4)-H1*X(5)-G1*X(6)+
1HN1*X(7)-GN1*X(8)-P/D
 G(7)=F(7,5)*X(5)+F(7,6)*X(6)+F(7,7)*X(7)+F(7,8)*X(8)-
1X(10)-SQRT(E)*(B12*RMX-B11*SMX)-QV*X(14)
 G(8)=F(8,5)*X(5)+F(8,6)*X(6)+F(8,7)*X(7)+F(8,8)*X(8)+
1E*(B11*RMX+B12*SMX)-QV
 G(9)=F(9,5)*X(5)+F(9,6)*X(6)+F(9,7)*X(7)+F(9,8)*X(8)+
1SQRT(E**3)*(B12*RMX-B11*SMX)
 G(10)=F(10,5)*X(5)+F(10,6)*X(6)+F(10,7)*X(7)+F(10,8)*X(8)+QK
 G(11)=F(11,1)*X(1)+F(11,2)*X(2)+F(11,3)*X(3)+F(11,4)*X(4)+QK
 G(12)=F(12,1)*X(1)+F(12,2)*X(2)+F(12,3)*X(3)+F(12,4)*X(4)-X(12)-
1SQRT(E)*(D12*RMZ-D11*SMZ)-QV*X(13)
```

```
G(13)=X(9)+X(10)*X(14)+B11*RMX+B12*SMX+CV*X(14)**2/2.
G(14)=X(11)+X(13)*X(12)+D11*RMZ+D12*SMZ+QV*X(13)**2/2.
F(1,13)=G(2)
F(12,13)=G(1)
F(7,14)=G(8)
F(8,14)=G(9)
RETURN
END
```

Typical Program for Part II (see Figure 9)

```
C     PVMT CAL. DEFL., MOMENTS AND STRESSES IN FULLY SUPPORTED SLAB
C     THE SLAB IS SUBJ TO MVNG LOADS AND HAS DIFF K-VALUES
C     ALL DEL GREATER THAN ZERO
C     WW AND SIGMX SHOULD BE DIMENSIONED ACCORDING TO NN
      DIMENSION A(12,13), INDEX(12), DEL(2), WW(500), SIGMA(500)
      DIMENSION AA(3),BA(3),BN(3),WO(3),VR(3),BT(3),AK(3),CR(3), AMO(3)
 5001 FORMAT (7F10.2)
 5020 READ (5,5001) (AK(I), I=1,2), (CR(I), I=1,2), ALEN, BH, AH
      AMU=0.15
      EY=4000000.
      RHO=.27100000E-02
      P=125.
      DO 80 KKK=1,4
      HCOUNT=KKK
      H=6.+(HCOUNT-1.)*2.
      QQ=H*(150.9/1728.)
      D=EY*H**3/(12.*(1.-AMU**2))
      QP=.10000000E+03
      IF (H .EQ. 6.) QP=.10000000E+08
 1001 FORMAT (4E18.8)
 1002 FORMAT (/)
 1003 FORMAT (8E14.6/8E14.6)
 1004 FORMAT (9E13.5)
      DO 91 KK=1,5
      VCOUNT=KK
      V=352.*(VCOUNT-1.)
      DO 92 I=1,2
      BT(I)=SQRT(SQRT(AK(I)/(4.*D)))
      WO(I)=P*BT(I)/(2.*AK(I))+QQ/AK(I)
      AMO(I)=P/(4.*BT(I))
      VR(I)=V/(SQRT(SQRT(4.*AK(I)*D/((RHO*H)**2))))
      DEL(I)=16.*(4.*(1.-CR(I)**2)*VR(I)**8-(8.-36.*CR(I)**2+
     127.*CR(I)**4)*VR(I)**4+4.)
      WRITE (6,1004) DEL(I),V, H
      IF (DEL(I)-0.) 81,81,93
   81 WRITE (6,1003) AK(1), AK(2), CR(1), CR(2)
      GO TO 84
   93 F=.10000000E+12
      B1=F
      B2=2.*VR(I)*VR(I)*BT(I)*BT(I)*F
      B3=(BT(I)**4)*((VR(I)**4)-1.)*F
      B4=(VR(I)**2)*(CR(I)**2)*F*(BT(I)**6)
      X=.02
      EPS=.10000000E-06
      CALL SITER (X,EPS,B1,B2,B3,B4)
      AA(I)=ABS(X)
      BN(I)=BT(I)*SQRT(2.*VR(I)**2+(AA(I)/BT(I))**2
     1+(2.*VR(I)*CR(I)*BT(I))/AA(I))
      BA(I)=BT(I)*SQRT(2.*VR(I)**2+(AA(I)/BT(I))**2
     1-(2.*VR(I)*CR(I)*BT(I))/AA(I))
   92 CONTINUE
      SIG=AMO(1)*6./(H**2)
      WRITE (6,1004) AK(1),AK(2),CR(1),CR(2),ALEN,AL1,BH,AH,X2
      WRITE (6,1003) AA(1),AA(2),BA(1),BA(2),BN(1),BN(2),VR(1),VR(2),
     1BT(1),BT(2),WO(1),WO(2),AMO(1),AMO(2), SIG
```

```
      AK(3)=AK(1)
      BA(3)=BA(1)
      AA(3)=AA(1)
      BN(3)=BN(1)
      G1=BA(1)**3-3.*BA(1)*AA(1)**2
      G2=BA(2)**3-3.*BA(2)*AA(2)**2
      G3=BA(3)**3-3.*BA(3)*AA(3)**2
      GN1=BN(1)**3-3.*BN(1)*AA(1)**2
      GN2=BN(2)**3-3.*BN(2)*AA(2)**2
      GN3=BN(3)**3-3.*BN(3)*AA(3)**2
      H1=3.*AA(1)*BA(1)**2-AA(1)**3
      H2=3.*AA(2)*BA(2)**2-AA(2)**3
      H3=3.*AA(3)*BA(3)**2-AA(3)**3
      HN1=3.*AA(1)*BN(1)**2-AA(1)**3
      HN2=3.*AA(2)*BN(2)**2-AA(2)**3
      HN3=3.*AA(3)*BN(3)**2-AA(3)**3
      Q1=2.*AA(1)*BA(1)
      Q2=2.*AA(2)*BA(2)
      Q3=2.*AA(3)*BA(3)
      QN1=2.*AA(1)*BN(1)
      QN2=2.*AA(2)*BN(2)
      QN3=2.*AA(3)*BN(3)
      F1=AA(1)**2-BA(1)**2
      F2=AA(2)**2-BA(2)**2
      F3=AA(3)**2-BA(3)**2
      FN1=AA(1)**2-BN(1)**2
      FN2=AA(2)**2-BN(2)**2
      FN3=AA(3)**2-BN(3)**2
      AL1=12.
      DO 99 KJK=1,3
      AL1=AL1*2.
      X2=-(12.+AL1)
      N=IFIX(1.+(2.*ALEN+AL1)/AH)
      NN=IFIX(1.-2.*X2/BH)
      DO 94 I=1,N
      SCOUNT=I
      S=ALEN-AH*(SCOUNT-1.)
      SM=S+AL1
      WRITE (6,1004) S, SM, AL1
      IF (S-0.) 199,100,100
  100 T1=EXP(AA(1)*S)
      T2=EXP(AA(2)*S)
      D1=EXP(-AA(1)*S)
      D2=EXP(-AA(2)*S)
      TM2=EXP(AA(2)*SM)
      DM2=EXP(-AA(2)*SM)
      DM3=EXP(-AA(3)*SM)
      R1=COS(BA(1)*S)
      S1=SIN(BA(1)*S)
      RN1=COS(BN(1)*S)
      SN1=SIN(BN(1)*S)
      R2=COS(BA(2)*S)
      S2=SIN(BA(2)*S)
      RN2=COS(BN(2)*S)
      SN2=SIN(BN(2)*S)
      RM2=COS(BA(2)*SM)
```

```
      SM2=SIN(BA(2)*SM)
      RNM2=COS(BN(2)*SM)
      SNM2=SIN(BN(2)*SM)
      RNM3=COS(BN(3)*SM)
      SNM3=SIN(BN(3)*SM)
      DO 101 IJ=1,12
      DO 101 JI=1,13
  101 A(IJ,JI)=0.
      A(1,4)=-1.
      A(1,6)=-1.
      A(2,3)=-BN(1)
      A(2,4)=AA(1)
      A(2,5)=-BA(1)
      A(2,6)=-AA(1)
      A(3,3)=QN1
      A(3,4)=-FN1
      A(3,5)=-Q1
      A(3,6)=-F1
      A(4,3)=-GN1
      A(4,4)=HN1
      A(4,5)=-G1
      A(4,6)=-H1
      DO 102 KI=5,6
      KJ=KI-4
      DO 102 KL=1,4
  102 A(KL,KJ)=-A(KL,KI)
      A(5,3)=D1*SN1
      A(5,4)=D1*RN1
      A(5,5)=T1*S1
      A(5,6)=T1*R1
      A(5,7)=-D2*SN2
      A(5,8)=-D2*RN2
      A(5,9)=-T2*S2
      A(5,10)=-T2*R2
      A(6,3)=D1*(BN(1)*RN1-AA(1)*SN1)
      A(6,4)=-D1*(AA(1)*RN1+BN(1)*SN1)
      A(6,5)=T1*(AA(1)*S1+BA(1)*R1)
      A(6,6)=T1*(AA(1)*R1-BA(1)*S1)
      A(6,7)=D2*(AA(2)*SN2-BN(2)*RN2)
      A(6,8)=D2*(AA(2)*RN2+BN(2)*SN2)
      A(6,9)=-T2*(AA(2)*S2+BA(2)*R2)
      A(6,10)=T2*(BA(2)*S2-AA(2)*R2)
      A(7,3)=D1*(FN1*SN1-QN1*RN1)
      A(7,4)=D1*(FN1*RN1+QN1*SN1)
      A(7,5)=T1*(F1*S1+Q1*R1)
      A(7,6)=T1*(F1*R1-Q1*S1)
      A(7,7)=D2*(QN2*RN2-FN2*SN2)
      A(7,8)=-D2*(FN2*RN2+QN2*SN2)
      A(7,9)=-T2*(F2*S2+Q2*R2)
      A(7,10)=T2*(Q2*S2-F2*R2)
      A(8,3)=D1*(HN1*SN1-GN1*RN1)
      A(8,4)=D1*(HN1*RN1+GN1*SN1)
      A(8,5)=-T1*(H1*S1+G1*R1)
      A(8,6)=T1*(G1*S1-H1*R1)
      A(8,7)=D2*(GN2*RN2-HN2*SN2)
      A(8,8)=-D2*(HN2*RN2+GN2*SN2)
```

```
      A(8,9)=T2*(G2*R2+H2*S2)
      A(8,10)=T2*(H2*R2-G2*S2)
      A(9,7)=DM2*SNM2
      A(9,8)=DM2*RNM2
      A(9,9)=TM2*SM2
      A(9,10)=TM2*RM2
      A(9,11)=-DM3*SNM3
      A(9,12)=-DM3*RNM3
      A(10,7)=DM2*(BN(2)*RNM2-AA(2)*SNM2)
      A(10,8)=-DM2*(AA(2)*RNM2+BN(2)*SNM2)
      A(10,9)=TM2*(AA(2)*SM2+BA(2)*RM2)
      A(10,10)=TM2*(AA(2)*RM2-BA(2)*SM2)
      A(10,11)=DM3*(AA(3)*SNM3-BN(3)*RNM3)
      A(10,12)=DM3*(AA(3)*RNM3+BN(3)*SNM3)
      A(11,7)=DM2*(FN2*SNM2-QN2*RNM2)
      A(11,8)=DM2*(FN2*RNM2+QN2*SNM2)
      A(11,9)=TM2*(F2*SM2+Q2*RM2)
      A(11,10)=TM2*(F2*RM2-Q2*SM2)
      A(11,11)=DM3*(QN3*RNM3-FN3*SNM3)
      A(11,12)=-DM3*(QN3*SNM3+FN3*RNM3)
      A(12,7)=DM2*(HN2*SNM2-GN2*RNM2)
      A(12,8)=DM2*(HN2*RNM2+GN2*SNM2)
      A(12,9)=-TM2*(H2*SM2+G2*RM2)
      A(12,10)=TM2*(G2*SM2-H2*RM2)
      A(12,11)=DM3*(GN3*RNM3-HN3*SNM3)
      A(12,12)=-DM3*(HN3*RNM3+GN3*SNM3)
      A(4,13)=P/D
      A(5,13)=QQ*(AK(1)-AK(2))/(AK(1)*AK(2))
      A(9,13)=QQ*(AK(2)-AK(3))/(AK(2)*AK(3))
      DO 411 KI=1,12
      DO 411 KL=1,13
  411 A(KI,KL)=QP*A(KI,KL)
      CALL CROUT (A,12,1,13,DETERM,INDEX)
      DO 103 J=1,NN
      XCOUNT=J
      Y=X2+(XCOUNT-1.)*BH
      IF (Y-S) 104,104,120
  104 RY=COS(BA(1)*Y)
      SY=SIN(BA(1)*Y)
      TY=EXP(AA(1)*Y)
      DY=EXP(-AA(1)*Y)
      RNY=COS(BN(1)*Y)
      SNY=SIN(BN(1)*Y)
      IF (Y-0.) 105,105,106
  105 W=TY*(A(1,1)*SY+A(2,1)*RY)+QQ/AK(1)
      AMOM=-D*TY*(A(1,1)*(F1*SY+Q1*RY)+A(2,1)*(F1*RY-Q1*SY))
      GO TO 501
  106 W=TY*(A(5,1)*SY+A(6,1)*RY)+DY*(A(3,1)*SNY+A(4,1)*RNY)+QQ/AK(1)
      AMOM=-D*(TY*(A(5,1)*(F1*SY+Q1*RY)+A(6,1)*(F1*RY-Q1*SY))+
     1DY*(A(3,1)*(FN1*SNY-QN1*RNY)+A(4,1)*(QN1*SNY+FN1*RNY)))
      GO TO 501
  120 IF (Y-SM) 121,131,131
  121 RY=COS(BA(2)*Y)
      SY=SIN(BA(2)*Y)
      TY=EXP(AA(2)*Y)
      DY=EXP(-AA(2)*Y)
```

```
      RNY=COS(BN(2)*Y)
      SNY=SIN(BN(2)*Y)
      W=TY*(A(9,1)*SY+A(10,1)*RY)+DY*(A(7,1)*SNY+A(8,1)*RNY)+CC/AK(2)
      AMCM=-C*(TY*(A(9,1)*(F2*SY+Q2*RY)+A(10,1)*(F2*RY-Q2*SY))+
     1DY*(A(7,1)*(FN2*SNY-QN2*RNY)+A(8,1)*(QN2*SNY+FN2*RNY)))
      GO TO 501
  131 RNY=COS(BN(3)*Y)
      SNY=SIN(BN(3)*Y)
      DY=EXP(-AA(3)*Y)
      W=DY*(A(11,1)*SNY+A(12,1)*RNY)+CQ/AK(3)
      AMCM=-D*DY*(A(11,1)*(FN3*SNY-QN3*RNY)+A(12,1)*(QN3*SNY+FN3*RNY))
  501 WW(J)=W
      SIGMA(J)=AMOM*6./(H**2)
  103 CONTINUE
      MAX=1
      WMAX=WW(1)
      DO 20 J=2,NN
      IF (WW(J)-WMAX) 20,20,10
   10 MAX=J
      WMAX=WW(J)
   20 CONTINUE
      RWMAX=WMAX/WO(1)
      YW=X2+FLOAT(MAX-1)*BH
      BETYW=BT(1)*YW
      WRITE (6,1001) YW, BETYW, WMAX, RWMAX
      MAX=1
      SIGMX=SIGMA(1)
      DO 30 J=2,NN
      IF (SIGMA(J)-SIGMX) 30,30,40
   40 MAX=J
      SIGMX=SIGMA(J)
   30 CONTINUE
      RSIGX=SIGMX/SIG
      YS=X2+FLOAT(MAX-1)*BH
      BETYS=BT(1)*YS
      WRITE (6,1001) YS, BETYS, SIGMX, RSIGX
      GO TO 94
  199 IF (SM-0.) 300,300,200
  200 T1=EXP(AA(1)*S)
      T2=EXP(AA(2)*S)
      D2=EXP(-AA(2)*S)
      TM2=EXP(AA(2)*SM)
      DM2=EXP(-AA(2)*SM)
      DM3=EXP(-AA(3)*SM)
      R1=COS(BA(1)*S)
      S1=SIN(BA(1)*S)
      R2=COS(BA(2)*S)
      S2=SIN(BA(2)*S)
      RN2=COS(BN(2)*S)
      SN2=SIN(BN(2)*S)
      RM2=COS(BA(2)*SM)
      SM2=SIN(BA(2)*SM)
      RNM2=COS(BN(2)*SM)
      SNM2=SIN(BN(2)*SM)
      RNM3=COS(BN(3)*SM)
      SNM3=SIN(BN(3)*SM)
```

```
      DO 201 IJ=1,12
      DO 201 JI=1,13
  201 A(IJ,JI)=0.
      A(1,4)=1.
      A(1,6)=1.
      A(2,3)=BA(2)
      A(2,4)=AA(2)
      A(2,5)=BN(2)
      A(2,6)=-AA(2)
      A(3,3)=Q2
      A(3,4)=F2
      A(3,5)=-QN2
      A(3,6)=FN2
      A(4,3)=G2
      A(4,4)=H2
      A(4,5)=GN2
      A(4,6)=-HN2
      DO 202 KI=3,6
      KJ=KI+4
      DO 202 KL=1,4
  202 A(KL,KJ)=-A(KL,KI)
      A(5,1)=T1*S1
      A(5,2)=T1*R1
      A(5,3)=-T2*S2
      A(5,4)=-T2*R2
      A(5,5)=-D2*SN2
      A(5,6)=-D2*RN2
      A(6,1)=T1*(AA(1)*S1+BA(1)*R1)
      A(6,2)=T1*(AA(1)*R1-BA(1)*S1)
      A(6,3)=-T2*(AA(2)*S2+BA(2)*R2)
      A(6,4)=T2*(BA(2)*S2-AA(2)*R2)
      A(6,5)=D2*(AA(2)*SN2-BN(2)*RN2)
      A(6,6)=D2*(AA(2)*RN2+BN(2)*SN2)
      A(7,1)=T1*(F1*S1+Q1*R1)
      A(7,2)=T1*(F1*R1-Q1*S1)
      A(7,3)=-T2*(F2*S2+Q2*R2)
      A(7,4)=-T2*(F2*R2-Q2*S2)
      A(7,5)=-D2*(FN2*SN2-QN2*RN2)
      A(7,6)=-D2*(FN2*RN2+QN2*SN2)
      A(8,1)=-T1*(H1*S1+G1*R1)
      A(8,2)=-T1*(H1*R1-G1*S1)
      A(8,3)=T2*(H2*S2+G2*R2)
      A(8,4)=T2*(H2*R2-G2*S2)
      A(8,5)=-D2*(HN2*SN2-GN2*RN2)
      A(8,6)=-D2*(HN2*RN2+GN2*SN2)
      A(9,7)=TM2*SM2
      A(9,8)=TM2*RM2
      A(9,9)=DM2*SNM2
      A(9,10)=DM2*RNM2
      A(9,11)=-DM3*SNM3
      A(9,12)=-DM3*RNM3
      A(10,7)=TM2*(AA(2)*SM2+BA(2)*RM2)
      A(10,8)=TM2*(AA(2)*RM2-BA(2)*SM2)
      A(10,9)=-DM2*(AA(2)*SNM2-BN(2)*RNM2)
      A(10,10)=-DM2*(AA(2)*RNM2+BN(2)*SNM2)
      A(10,11)=DM3*(AA(3)*SNM3-BN(3)*RNM3)
```

```
      A(10,12)=DM3*(AA(3)*RNM3+BN(3)*SNM3)
      A(11,7)=TM2*(F2*SM2+Q2*RM2)
      A(11,8)=TM2*(F2*RM2-Q2*SM2)
      A(11,9)=DM2*(FN2*SNM2-QN2*RNM2)
      A(11,10)=DM2*(FN2*RNM2+QN2*SNM2)
      A(11,11)=-DM3*(FN3*SNM3-QN3*RNM3)
      A(11,12)=-DM3*(FN3*RNM3+QN3*SNM3)
      A(12,7)=-TM2*(H2*SM2+G2*RM2)
      A(12,8)=-TM2*(H2*RM2-G2*SM2)
      A(12,9)=DM2*(HN2*SNM2-GN2*RNM2)
      A(12,10)=DM2*(HN2*RNM2+GN2*SNM2)
      A(12,11)=-DM3*(HN3*SNM3-GN3*RNM3)
      A(12,12)=-DM3*(HN3*RNM3+GN3*SNM3)
      A(4,13)=P/D
      A(5,13)=QQ*(AK(1)-AK(2))/(AK(1)*AK(2))
      A(9,13)=QQ*(AK(2)-AK(3))/(AK(2)*AK(3))
      DO 422 KI=1,12
      DO 422 KL=1,13
  422 A(KI,KL)=QP*A(KI,KL)
      CALL CROUT (A,12,1,13,DETERM,INDEX)
      DO 203 J=1,NN
      XCOUNT=J
      Y=X2+(XCOUNT-1.)*BH
      IF (Y-S) 204,204,210
  204 RY=COS(BA(1)*Y)
      SY=SIN(BA(1)*Y)
      TY=EXP(AA(1)*Y)
      W=TY*(A(1,1)*SY+A(2,1)*RY)+QQ/AK(1)
      AMOM=-D*TY*(A(1,1)*(F1*SY+Q1*RY)+A(2,1)*(F1*RY-Q1*SY))
      GO TO 502
  210 IF (Y-SM) 211,214,214
  211 RY=COS(BA(2)*Y)
      SY=SIN(BA(2)*Y)
      TY=EXP(AA(2)*Y)
      DY=EXP(-AA(2)*Y)
      RNY=COS(BN(2)*Y)
      SNY=SIN(BN(2)*Y)
      IF (Y-0.) 212,212,213
  212 W=TY*(A(3,1)*SY+A(4,1)*RY)+DY*(A(5,1)*SNY+A(6,1)*RNY)+QQ/AK(2)
      AMOM=-D*(TY*(A(4,1)*(F2*RY-Q2*SY)+A(3,1)*(F2*SY+Q2*RY))+
     1DY*( A(6,1)*(QN2*SNY+FN2*RNY)+A(5,1)*(FN2*SNY-QN2*RNY)))
      GO TO 502
  213 W=TY*(A(7,1)*SY+A(8,1)*RY)+DY*(A(9,1)*SNY+A(10,1)*RNY)+QQ/AK(2)
      AMOM=-D*(TY*(A(8,1)*(F2*RY-Q2*SY)+A(7,1)*(F2*SY+Q2*RY))+
     1DY*(A(10,1)*(QN2*SNY+FN2*RNY)+A(9,1)*(FN2*SNY-QN2*RNY)))
      GO TO 502
  214 RNY=COS(BN(3)*Y)
      SNY=SIN(BN(3)*Y)
      DY=EXP(-AA(3)*Y)
      W=DY*(A(11,1)*SNY+A(12,1)*RNY)+QQ/AK(3)
      AMOM=-D*DY*(A(11,1)*(FN3*SNY-QN3*RNY)+A(12,1)*(QN3*SNY+FN3*RNY))
  502 WW(J)=W
      SIGMA(J)=AMOM*6./(H**2)
  203 CONTINUE
      MAX=1
      WMAX=WW(1)
```

```
      DO 22 J=2,NN
      IF (WW(J)-WMAX) 22,22,12
12    MAX=J
      WMAX=WW(J)
22    CONTINUE
      RWMAX=WMAX/WO(1)
      YW=X2+FLOAT(MAX-1)*BH
      BETYW=BT(1)*YW
      WRITE (6,1001) YW, BETYW, WMAX, RWMAX
      MAX=1
      SIGMX=SIGMA(1)
      DO 32 J=2,NN
      IF (SIGMA(J)-SIGMX) 32,32,42
42    MAX=J
      SIGMX=SIGMA(J)
32    CONTINUE
      RSIGX=SIGMX/SIG
      YS=X2+FLOAT(MAX-1)*BH
      BETYS=BT(1)*YS
      WRITE (6,1001) YS, BETYS, SIGMX, RSIGX
      GO TO 94
300   T1=EXP(AA(1)*S)
      T2=EXP(AA(2)*S)
      D2=EXP(-AA(2)*S)
      TM2=EXP(AA(2)*SM)
      DM2=EXP(-AA(2)*SM)
      TM3=EXP(AA(3)*SM)
      DM3=EXP(-AA(3)*SM)
      R1=COS(BA(1)*S)
      S1=SIN(BA(1)*S)
      R2=COS(BA(2)*S)
      S2=SIN(BA(2)*S)
      RN2=COS(BN(2)*S)
      SN2=SIN(BN(2)*S)
      RM2=COS(BA(2)*SM)
      SM2=SIN(BA(2)*SM)
      RNM2=COS(BN(2)*SM)
      SNM2=SIN(BN(2)*SM)
      RM3=COS(BA(3)*SM)
      SM3=SIN(BA(3)*SM)
      RNM3=COS(BN(3)*SM)
      SNM3=SIN(BN(3)*SM)
      DO 301 IJ=1,12
      DO 301 JI=1,13
301   A(IJ,JI)=0.
      A(1,8)=1.
      A(1,10)=1.
      A(2,7)=BA(3)
      A(2,8)=AA(3)
      A(2,9)=BN(3)
      A(2,10)=-AA(3)
      A(3,7)=Q3
      A(3,8)=F3
      A(3,9)=-QN3
      A(3,10)=FN3
      A(4,7)=G3
```

```
      A(4,8)=H3
      A(4,9)=GN3
      A(4,10)=-HN3
      DO 302 KI=9,10
      KJ=KI+2
      DO 302 KL=1,4
  302 A(KL,KJ)=-A(KL,KI)
      A(5,1)=T1*S1
      A(5,2)=T1*R1
      A(5,3)=-T2*S2
      A(5,4)=-T2*R2
      A(5,5)=-D2*SN2
      A(5,6)=-D2*RN2
      A(6,1)=T1*(AA(1)*S1+BA(1)*R1)
      A(6,2)=T1*(AA(1)*R1-BA(1)*S1)
      A(6,3)=-T2*(AA(2)*S2+BA(2)*R2)
      A(6,4)=-T2*(AA(2)*R2-BA(2)*S2)
      A(6,5)=-D2*(BN(2)*RN2-AA(2)*SN2)
      A(6,6)=D2*(AA(2)*RN2+BN(2)*SN2)
      A(7,1)=T1*(Q1*R1+F1*S1)
      A(7,2)=T1*(F1*R1-Q1*S1)
      A(7,3)=-T2*(Q2*R2+F2*S2)
      A(7,4)=-T2*(F2*R2-Q2*S2)
      A(7,5)=-D2*(FN2*SN2-QN2*RN2)
      A(7,6)=-D2*(QN2*SN2+FN2*RN2)
      A(8,1)=-T1*(H1*S1+G1*R1)
      A(8,2)=-T1*(H1*R1-G1*S1)
      A(8,3)=T2*(H2*S2+G2*R2)
      A(8,4)=T2*(H2*R2-G2*S2)
      A(8,5)=-D2*(HN2*SN2-GN2*RN2)
      A(8,6)=-D2*(HN2*RN2+GN2*SN2)
      A(9,3)=TM2*SM2
      A(9,4)=TM2*RM2
      A(9,5)=DM2*SNM2
      A(9,6)=DM2*RNM2
      A(9,7)=-TM3*SM3
      A(9,8)=-TM3*RM3
      A(9,9)=-DM3*SNM3
      A(9,10)=-DM3*RNM3
      A(10,3)=TM2*(AA(2)*SM2+BA(2)*RM2)
      A(10,4)=TM2*(AA(2)*RM2-BA(2)*SM2)
      A(10,5)=DM2*(BN(2)*RNM2-AA(2)*SNM2)
      A(10,6)=-DM2*(AA(2)*RNM2+BN(2)*SNM2)
      A(10,7)=-TM3*(AA(3)*SM3+BA(3)*RM3)
      A(10,8)=-TM3*(AA(3)*RM3-BA(3)*SM3)
      A(10,9)=-DM3*(BN(3)*RNM3-AA(3)*SNM3)
      A(10,10)=DM3*(AA(3)*RNM3+BN(3)*SNM3)
      A(11,3)=TM2*(Q2*RM2+F2*SM2)
      A(11,4)=TM2*(F2*RM2-Q2*SM2)
      A(11,5)=DM2*(FN2*SNM2-QN2*RNM2)
      A(11,6)=DM2*(QN2*SNM2+FN2*RNM2)
      A(11,7)=-TM3*(Q3*RM3+F3*SM3)
      A(11,8)=-TM3*(F3*RM3-Q3*SM3)
      A(11,9)=-DM3*(FN3*SNM3-QN3*RNM3)
      A(11,10)=-DM3*(QN3*SNM3+FN3*RNM3)
      A(12,3)=-TM2*(H2*SM2+G2*RM2)
```

```
      A(12,4)=TM2*(G2*SM2-H2*RM2)
      A(12,5)=-DM2*(GN2*RNM2-HN2*SNM2)
      A(12,6)=DM2*(HN2*RNM2+GN2*SNM2)
      A(12,7)=TM3*(H3*SM3+G3*RM3)
      A(12,8)=-TM3*(G3*SM3-P3*RM3)
      A(12,9)=DM3*(GN3*RNM3-HN3*SNM3)
      A(12,10)=-DM3*(HN3*RNM3+GN3*SNM3)
      A(4,13)=P/D
      A(5,13)=QQ*(AK(1)-AK(2))/(AK(1)*AK(2))
      A(9,13)=QQ*(AK(2)-AK(3))/(AK(2)*AK(3))
      DO 433 KI=1,12
      DO 433 KL=1,13
433   A(KI,KL)=QP*A(KI,KL)
      CALL CROUT (A,12,1,13,DETERM,INDEX)
      DO 303 J=1,NN
      XCOUNT=J
      Y=X2+(XCOUNT-1.)*BH
      IF (Y-S) 304, 304, 305
304   RY=COS(BA(1)*Y)
      SY=SIN(BA(1)*Y)
      TY=EXP(AA(1)*Y)
      W=TY*(A(1,1)*SY+A(2,1)*RY)+QQ/AK(1)
      AMOM=-D*TY*(A(1,1)*(F1*SY+Q1*RY)+A(2,1)*(F1*RY-Q1*SY))
      GO TO 503
305   IF (Y-SM) 306,306,307
306   RY=COS(BA(2)*Y)
      SY=SIN(BA(2)*Y)
      TY=EXP(AA(2)*Y)
      DY=EXP(-AA(2)*Y)
      RNY=COS(BN(2)*Y)
      SNY=SIN(BN(2)*Y)
      W=TY*(A(3,1)*SY+A(4,1)*RY)+DY*(A(5,1)*SNY+A(6,1)*RNY)+QQ/AK(2)
      AMOM=-D*(TY*(A(4,1)*(F2*RY-Q2*SY)+A(3,1)*(F2*SY+Q2*RY))+
     1DY*( A(6,1)*(QN2*SNY+FN2*RNY)+A(5,1)*(FN2*SNY-QN2*RNY)))
      GO TO 503
307   RY=COS(BA(3)*Y)
      SY=SIN(BA(3)*Y)
      TY=EXP(AA(3)*Y)
      DY=EXP(-AA(3)*Y)
      SNY=SIN(BN(3)*Y)
      RNY=COS(BN(3)*Y)
      IF (Y-0.) 308,309,309
308   W=TY*(A(7,1)*SY+A(8,1)*RY)+DY*(A(9,1)*SNY+A(10,1)*RNY)+QQ/AK(3)
      AMOM=-D*(TY*(A(8,1)*(F3*RY-Q3*SY)+A(7,1)*(F3*SY+Q3*RY))+
     1DY*(A(10,1)*(QN3*SNY+FN3*RNY)+A(9,1)*(FN3*SNY-QN3*RNY)))
      GO TO 503
309   W=DY*(A(11,1)*SNY+A(12,1)*RNY)+QQ/AK(3)
      AMOM=-D*DY*(A(11,1)*(FN3*SNY-QN3*RNY)+A(12,1)*(QN3*SNY+FN3*RNY))
503   WW(J)=W
      SIGMA(J)=AMOM*6./(H**2)
303   CONTINUE
      MAX=1
      WMAX=WW(1)
      DO 21 J=2,NN
      IF (WW(J)-WMAX) 21,21,11
11    MAX=J
```

```
      WMAX=WW(J)
 21   CONTINUE
      RWMAX=WMAX/WO(1)
      YW=X2+FLOAT(MAX-1)*BH
      BETYW=BT(1)*YW
      WRITE (6,1001) YW, BETYW, WMAX, RWMAX
      MAX=1
      SIGMX=SIGMA(1)
      DO 31 J=2,NN
      IF (SIGMA(J)-SIGMX) 31,31,41
 41   MAX=J
      SIGMX=SIGMA(J)
 31   CONTINUE
      RSIGX=SIGMX/SIG
      YS=X2+FLOAT(MAX-1)*BH
      BETYS=BT(1)*YS
      WRITE (6,1001) YS, BETYS, SIGMX, RSIGX
 94   CONTINUE
      WRITE (6,1002)
 99   CONTINUE
      WRITE (6,1002)
 91   CONTINUE
84    WRITE (6,1002)
 80   CONTINUE
 95   GO TO 5020
      END
```

```
      SUBROUTINE SITER (X,EPS,B1,B2,B3,B4)
 24   XNB=X
      XN1=B1*XNB**6+B2*XNB**4+B3*XNB**2-B4
      XN2=6.*B1*XNB**5+4.*B2*XNB**3+2.*B3*XNB
      X=XNB-XN1/XN2
      IF (ABS(X-XNB)-EPS) 26,26,25
 25   GO TO 24
 26   RETURN
      END
```

```
C      CRCUT REDUCTION
       SUBROUTINE CROUT(A,N,M,NN,DETERM,INDEX)
       DIMENSION A(N,NN),INDEX(N)
       DET=1.0
       JZ=N-1
       JA=N+1
       DO 30 I=1,N
    30 INDEX(I)=I
       DO 700 J=1,NN
       DO 800 II=1,N
       SUM=0.0
       I=INDEX(II)
       IF(II-J)33,34,34
    33 IF(II-1) 9000,9200,9000
  9000 LLLL=II-1
       DO 9100 K=1,LLLL
       IPPP=INDEX(K)
  9100 SUM=SUM+A(I,K)*A(IPPP,J)
  9200 A(I,J)=(A(I,J)-SUM)/A(I,II)
       GO TO 800
    34 IF(J-1) 8000,8200,8000
  8000 LLLL=J-1
       DO 8100 K=1,LLLL
       IPPP=INDEX(K)
  8100 SUM=SUM+A(I,K)*A(IPPP,J)
  8200 A(I,J)=A(I,J)-SUM
   800 CONTINUE
       IF(J-N)41,700,700
    41 L=INDEX(J)
       KA=L
       HIGH=A(L,J)
       KZ=0
       DO 35 I=J,JZ
       JC=I+1
       L=INDEX(JC)
       IF(ABS (HIGH)-ABS (A(L,J)))36,35,35
    36 HIGH=A(L,J)
       KA=L
       KZ=1
    35 CONTINUE
       IF(KZ.NE. 0)  DET=-DET
       IF(ABS (HIGH)-1.E-05)31,31,3200
    31 WRITE(6,32) HIGH
    32 FORMAT(48HOTHE PIVOT ELEMENT IS LESS THAN 1.E-05 VALUE IS E20.8)
  3200 DO 37 K=1,N
       KK=K
       IF(INDEX(K)-KA)37,38,37
    37 CONTINUE
    38 ITEMP=INDEX(J)
       INDEX(J)=INDEX(KK)
       INDEX(KK)=ITEMP
   700 CONTINUE
       IF(M)2000,1000,2000
  2000 L=N-1
       DO 39 J=JA,NN
```

```
      LL=1
      DO 42 K=1,N
      IF(ABS (A(K,J))-0.0)43,42,43
   42 CONTINUE
      IZ=INDEX(N)
      IF(ABS (A(IZ,N))-1.0E-02)46,46,44
   44 WRITE(6,45)
   45 FORMAT(59H0                                   ONLY SOLUTION IS ZERO VEC
     1TOR)
      GO TO 10
   46 A(IZ,J)=5.0000
      IZZ=INDEX(N-1)
      IF(ABS (A(IZZ,N))-1.0E-04)47,47,43
   47 A(IZZ,J)=2.50000
      LL=2
   43 DO 40 IJ=LL,L
      SUM1=0.0
      II=N-IJ
      I=INDEX(II)
      LL=II+1
      DO 9300 K=LL,N
      IP=INDEX(K)
 9300 SUM1=SUM1+A(I,K)*A(IP,J)
      A(I,J)=A(I,J)-SUM1
   40 CONTINUE
   39 CONTINUE
 1000 DETERM=1.0
      DO 900 I=1,N
      K=INDEX(I)
  900 DETERM=DETERM*A(K,I)
      DETERM=DETERM*DET
      DO 400 I=1,N
      DO 400 J=JA,NN
      K=INDEX(I)
      L=J-N
  400 A(I,L)=A(K,J)
   10 RETURN
      END
```

# APPENDIX IV

APPENDIX IV

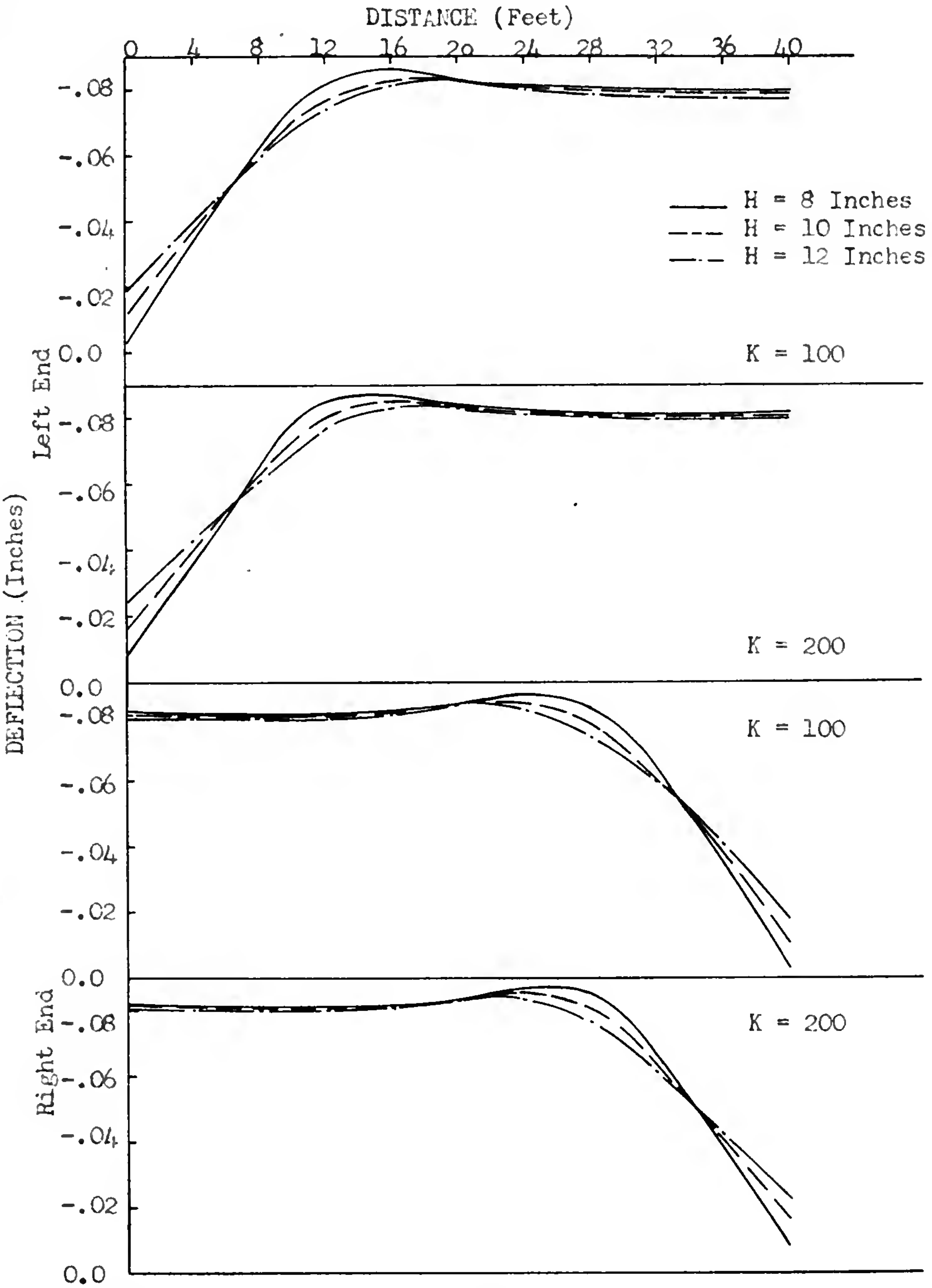


Figure IV-1 Deflection at Ends of Slab versus Position of Moving Load ( $C_r$=1.5, ΔT=30 Deg., v=0 mph.)

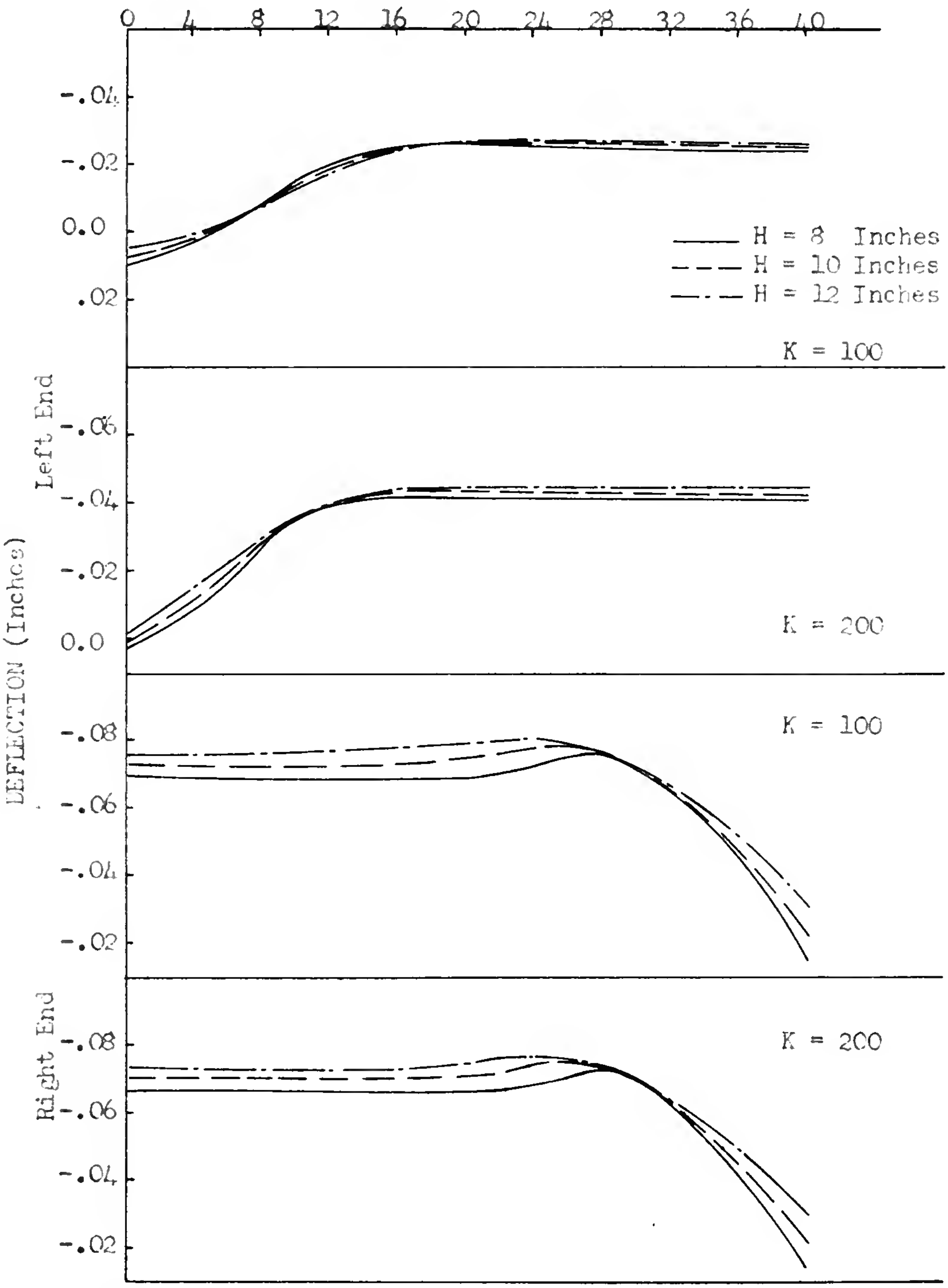


Figure IV-2 Deflection at Ends of Slab versus Position of Moving Load ( $C_r$=1.5, $\Delta T$=30 Deg., v=40 mph.)

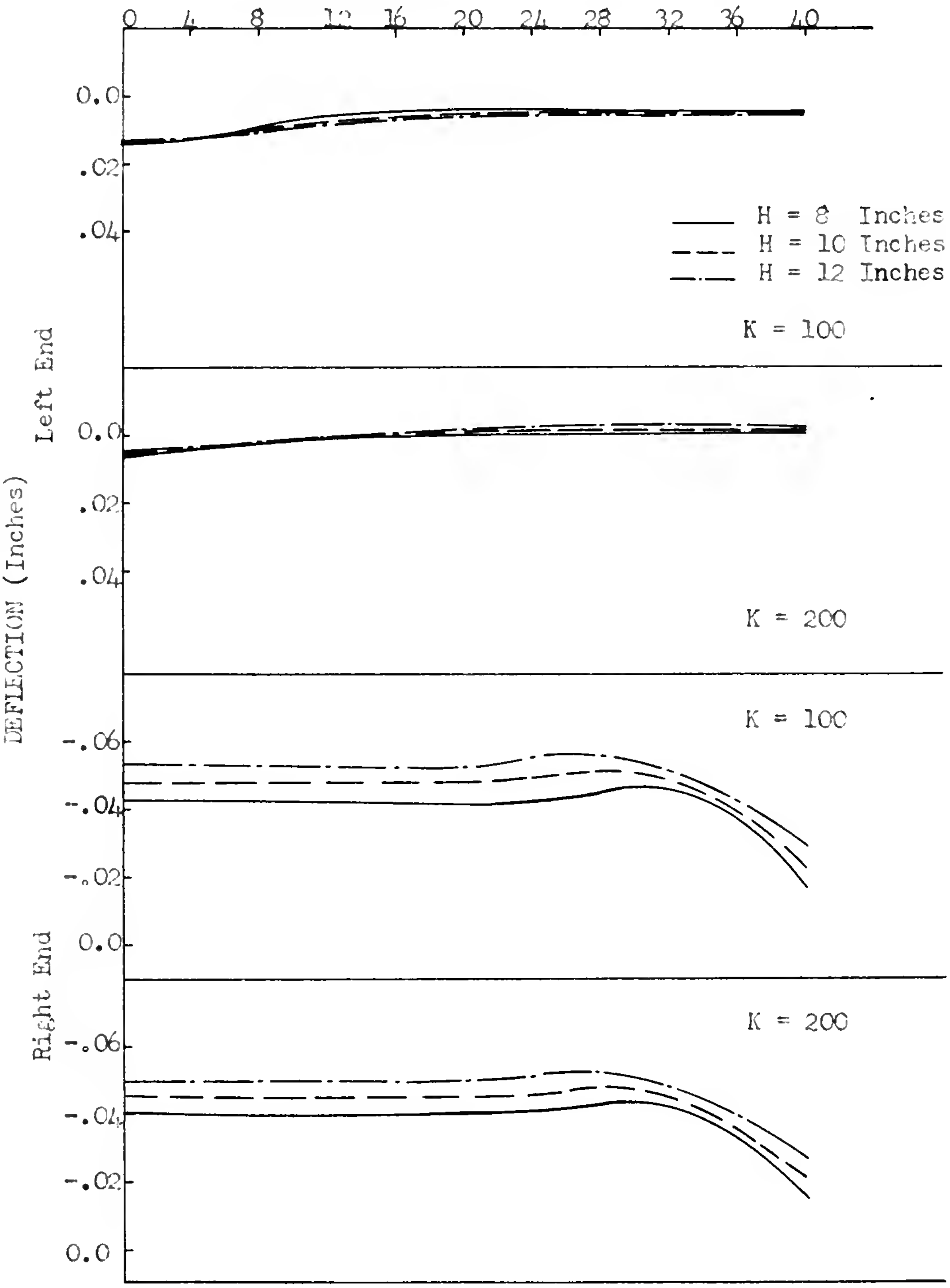


Figure IV-3 Deflection at Ends of Slab versus Position of Moving Load ( $C_r$=1.5, $\Delta$T=30 Deg., v=80 mph.)

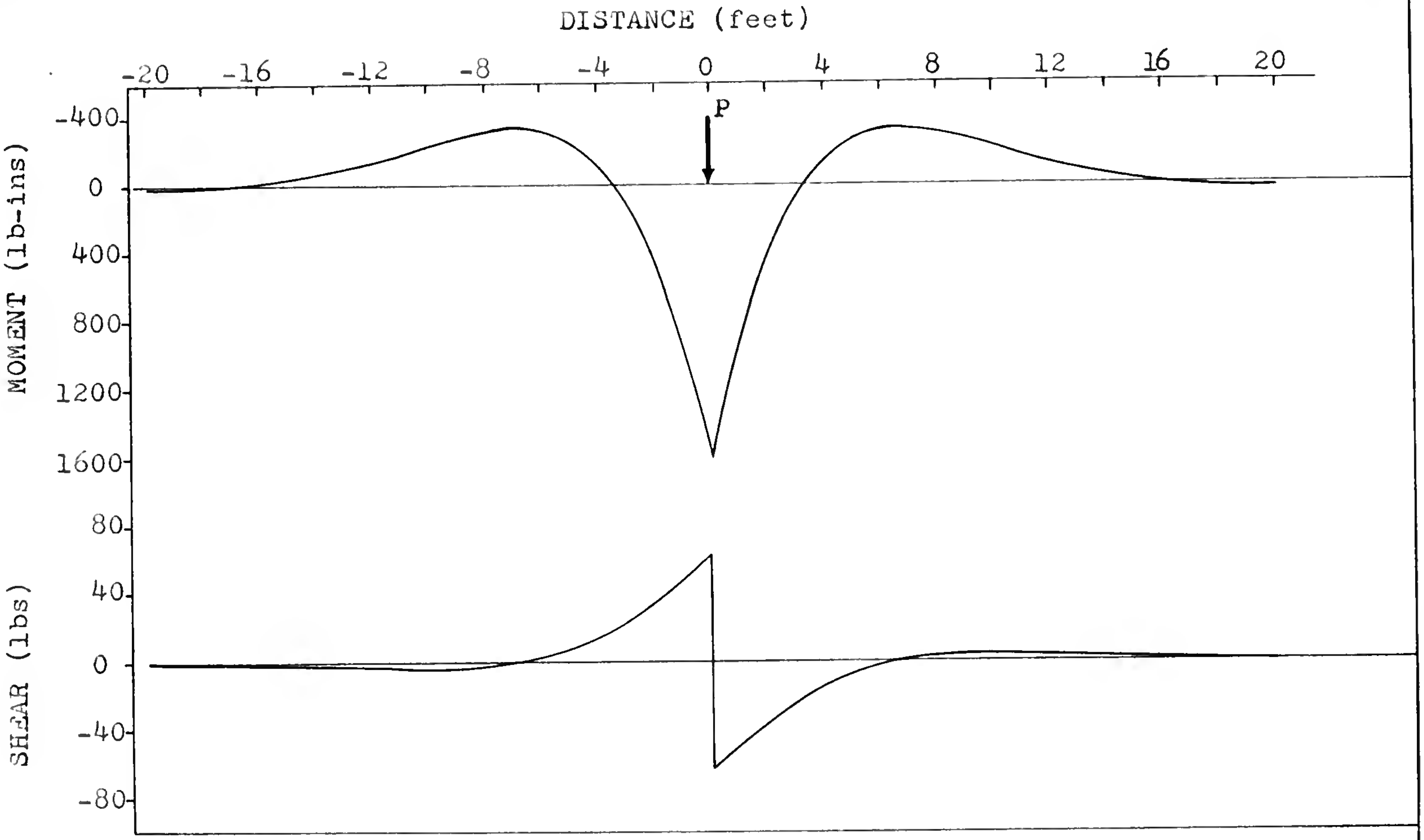


Figure IV-4 Shear and Moment Diagrams for Infinite Slab
($K_o$ = 100 pci., H = 8 inches, v = 0 mph.)

VITA

# VITA

Karl H. Lewis was born in St. Michael, Barbados, West Indies on January 15, 1936 and received his early education in the British School System. He is a citizen of Barbados, West Indies.

In June 1953, after graduation from Combermere High School, he joined the staff of Cable and Wireless Ltd., Barbados, West Indies. After spending four years as a Wireless Telegraphist, he resigned his position and entered Howard University where he received his BSCE with honors in 1961.

In September 1961 he joined the staff of Purdue University as a Graduate Assistant and received the MSCE in August 1963. That same year (1963) he became a Teaching Assistant in the School of Civil Engineering.

Mr. Lewis is now an Assistant Professor of Civil Engineering at the University of Pittsburgh. He is a member of Sigma Xi, Tau Beta Pi, Pi Mu Epsilon, AIEE, and an associate member of ASCE. He is married and has no children.